

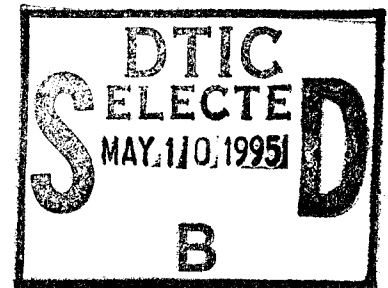
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Technical Report No. 1

THEORY OF ACOUSTIC RADIATION PRESSURE

By

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July 25, 1951

Submitted By

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ABSTRACT

A detailed study is presented of the acoustic radiation pressure exerted by plane compressional waves in non-viscous liquids and gases upon a plane obstacle. The present report is largely a further development and extension of a very comprehensive and penetrating treatment of acoustic radiation pressure by L. Brillouin. The theory of plane waves in liquids is extended to the case of finite amplitudes. From this more general point of view the effects at small amplitudes are derived and discussed in detail. Formulas are given for normal and oblique incidence of the acoustic beam for small amplitudes, valid for any reflection coefficient of the receiving plane. The radiation pressure at finite amplitudes upon a perfect absorber is calculated.

Special consideration is devoted to the actual physical processes involved; the meaning of the so-called Rayleigh pressure and Langevin pressure is discussed. For gases the radiation pressure as well as the Rayleigh pressure in progressive and standing waves are computed.

The report concludes with an assembly of the chief equations.

a	= space coordinate
C	= constant of integration
c	= phase velocity of light or sound
c_p	= specific heat at constant pressure
\bar{E}	= time-average value of the density of <u>total</u> energy for a harmonic wave traveling in one direction ($\bar{E} = \rho_0 \omega^2 \xi_0^2 / 2$)
\bar{E}_{kin}	= time-average of the density of acoustic kinetic energy
\bar{E}_{pot}	= time-average of the density of acoustic potential energy
\vec{E}	= electric field strength
f	= frequency
\vec{H}	= magnetic field strength
h	= width of surface receiving acoustic radiation (Fig. 12)
J	= intensity (power per unit area) of acoustic radiation per square centimeter
$J_1(x)$	= Bessel function of order 1 and argument x
k	= $2\pi/\lambda = \omega/c$
l	= length of surface receiving acoustic radiation (Fig. 12)
m	= mass
P_{rad}, P_x	= radiation pressure or its component in the positive x-direction
p_0	= hydrostatic pressure
p	= excess pressure due to compressional wave motion
p_t	= $p_0 + p$ = total pressure
S	= shearing stress due to radiation pressure at oblique incidence
T	= stress tensor, also absolute temperature
T_{ik}	= components of stress tensor
T_p	= $1/f$ = period
t	= time coordinate
u_x, u_y, u_z	= particle velocities in the x-, y-, z-directions

- u = particle velocity in a plane compressional wave along the x-direction ($u = u_x$)
 x, y, z = space coordinates
 x', y' = space coordinates of a system rotated through an angle ϑ (Fig. 5)
 V = unit volume; ΔV = small change in V
 V_0 = undisturbed unit volume

 β = compressibility of liquid (cm^2 per unit force)
 γ = amplitude reflection coefficient ($\gamma \leq 1$)
 γ_c = c_p/c_v = ratio of specific heats at constant pressure (c_p) and constant volume (c_v)
 Δ = ($\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$)
 δ = amplitude transmission coefficient ($\delta \leq 1$)
 ϵ = dielectric permittivity
 ϵ_{ik} = components of strain tensor (see Eq. (2.7))
 η = viscosity coefficient
 θ = phase angle of reflected wave
 ϑ = angle between the normal to a receiving plane and the direction x of the wave propagation
 λ = wavelength
 μ = magnetic permeability
 ρ = density; ρ_0 = undisturbed density
 σ = entropy
 τ_{\pm} = $\omega t \pm kx$
 ξ, η, ζ = particle displacements in the x, y, z -directions
 ω = $2\pi f$ = angular frequency

Note

One bar over a symbol denotes the time-average value of the quantity concerned, as \bar{p} , two bars the average value in time and space, as $\bar{\bar{p}}$.

A star * indicates that the quantity concerned refers to a moving particle or volume element (Lagrangian coordinates), as p^* .

PART I

THE CONCEPT OF RADIATION PRESSURE

1. Radiation Pressure in Electrodynamics

The concept of radiation pressure originated in electrodynamics. If we consider a plane surface emitting a plane electromagnetic wave (Fig. 1)

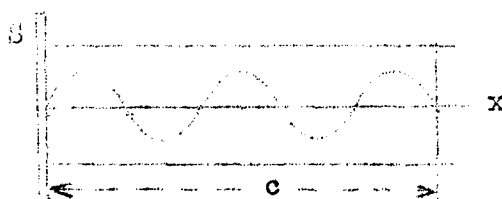


Fig. 1

Plane electromagnetic wave emitted
by a plane surface

in the positive x -direction, a reacting force is exerted upon the emitting plane due to the transport of momentum by the electromagnetic field in the direction of wave propagation. Imagine a cylinder with a cross-section of 1 cm^2 perpendicular to the x -axis. A wave front leaving the emitting surface at any time t reaches a cross-section of this cylinder at the distance c one second later, where c is the velocity of the wave. Assuming, for simplicity, an electromagnetic wave of a rather high frequency, so great a number of wavelengths may fill the distance c that we can speak of a mean total electromagnetic energy averaged over the length c of our cylinder. Dividing this mean total energy by c we obtain the mean energy-density \bar{E} , that is, the mean energy per cm^3 of the electromagnetic wave. If the emission is steady, \bar{E} is independent of time.

The whole energy filling the cylinder of length c , and therefore leaving the emitting surface each second, is $\bar{E} \cdot c$. According to the equivalence of mass and energy this corresponds to a mass transport in the positive x -direction. Let the whole mass filling the cylinder of length c be $m = \rho c$, ρ denoting the electromagnetic "mass-density" per cm^3 . Then we have

$$mc^2 = (\rho c) \cdot c^2 = \bar{E}c$$

and

$$\rho = \frac{\bar{E}}{c^2} \quad (1.1)$$

The equivalent mechanical momentum emitted each second from each cm^2 of the radiating surface is $m \cdot c = (\rho c) \cdot c = \rho c^2 = \bar{E}$. According to Newton's Law a reaction-force is exerted on each square centimeter of the radiating surface, which equals the momentum $mc = \bar{E}$ contributed per second to the emitted wave. This reacting force P_x per cm^2 in the negative x -direction is numerically equal to the so-called electromagnetic radiation pressure P_{rad} :

$$\underline{P_x = mc = P_{\text{rad}} = \bar{E}} \quad (1.2)$$

P_{rad} equals the mean electromagnetic density averaged over a wavelength in space:

$$P_{\text{rad}} = \bar{E} = \frac{1}{\lambda} \int_a^{a+\lambda} \left(\epsilon \frac{\vec{E}^2}{2} + \mu \frac{\vec{H}^2}{2} \right) dx \quad (1.3)$$

where $\lambda = c/f = \text{wavelength}$; ϵ is the permittivity; μ is the permeability; \vec{E} , \vec{H} are the electric and magnetic field vectors of the plane wave.

If the emitting surface were free to move, this radiation-force would cause a motion of the surface in the direction opposite to that of the emitted radiation, in such a way that the mass-center of the whole system (emitting mass + mass-equivalent of radiated energy) remained fixed in space.

Now we consider the case in which the emitted radiation strikes a plane totally absorbing surface perpendicular to the x-axis. Such a surface is usually called a surface of a black body. The electromagnetic momentum which one cm^2 of the black surface absorbs each second amounts likewise to $mc = \rho c^2 = \bar{E}$. This gain in momentum per second corresponds to a radiation force on each cm^2 of the receiving area in the positive x-direction. So we say that the electromagnetic radiation exerts a radiation pressure upon each square centimeter of the absorbing surface, which equals the mean energy density \bar{E} of the electromagnetic wave.

If the surface perpendicular to the direction of motion of the electromagnetic wave is not black, but totally reflecting, the momentum of the wave changes its sign at the reflector from $+mc$ to $-mc$. So the total change in momentum per second and per cm^2 of the wave amounts to $2mc = 2\rho c^2 = 2\bar{E}$. Therefore the radiation pressure P_{rad} on a perfectly reflecting surface is equal to $2\bar{E}$. If the receiving surface partially absorbs and partially reflects, the radiation pressure has a value between \bar{E} and $2\bar{E}$, depending on the coefficient of the reflection of the surface, which expresses the ratio of the reflected to the incident wave energy.

2. The Concept of Pressure in Fluids

In view of our later treatment of acoustic radiation

pressure, we first consider the meaning of the word "pressure". In the physics of deformable bodies (solids, liquids, gases) the word pressure is exemplified by the hydrodynamic pressure in a fluid. The pressure is a scalar $p(x,y,z)$ which may change from point to point in space. If we consider a volume-element of arbitrary shape inside a fluid, the hydrodynamic pressure p exerts the same force in all directions if the volume is so small that the change in pressure with respect to x,y,z can be neglected (Fig. 2).

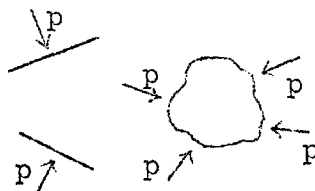


Fig. 2

Hydrostatic pressure p in fluids

Or, if we imagine a small surface element in the fluid, the pressure p on it is the same for each orientation in space and always normal to the surface.

Instead of using the word pressure we can also say that there is a stress T acting on any small element of area perpendicular to the element and of the same value for every orientation in space of the element. We write

$$T = - p \quad (1.4)$$

since usually the pressure is called positive if a volume element is compressed, while a stress is called positive when tensile (Fig. 3).

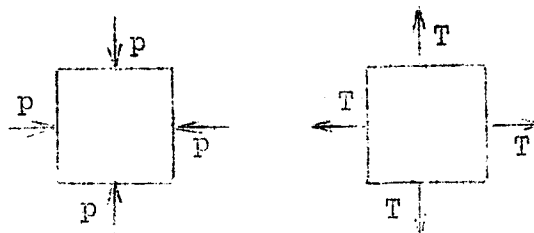


Fig. 3

Directions of positive pressure p
and positive uniform stress T

At this point it is desirable to consider the general expression for a stress-tensor. This expression will then be specialized for the case of a fluid. Any stress, whether due to mechanical forces or to an electromagnetic field, is a tensor having in general nine components. It is represented by

$$T_{ik} = \begin{vmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{vmatrix} \quad (1.5)$$

The T_{ii} (T_{xx} , T_{yy} , T_{zz}) represent the normal forces perpendicular to surface elements in the yz -, xz -, and xy -planes, whereas the shearing forces T_{ik} (T_{xy} , T_{xz} , ...) act parallel to these surface elements. In fluids under mechanical action, disregarding viscous forces, only the T_{ii} exist and the stress tensor reduces to

$$T_{liq} = \begin{vmatrix} T_{xx} & 0 & 0 \\ 0 & T_{yy} & 0 \\ 0 & 0 & T_{zz} \end{vmatrix} \quad (1.6)$$

Furthermore, as we have stated, the force on a surface element in a fluid is independent of its orientation in space, so that

$$T_{xx} = T_{yy} = T_{zz} = T; \text{ or}$$

$$T_{liq} = \begin{vmatrix} T & 0 & 0 \\ 0 & T & 0 \\ 0 & 0 & T \end{vmatrix} = \begin{vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{vmatrix} \quad (1.7)$$

According to the rules of tensor calculus the quantity

$$I = T_{xx} + T_{yy} + T_{zz} \quad (1.8)$$

is invariant; that is, it has the same value for any set of rectangular coordinates at the same point. In our case $T_{xx} = T_{yy} = T_{zz} = -p$, and the quantity $I = -3p$, or

$$-p = \frac{I}{3} \quad (1.9)$$

The hydrostatic mean pressure p , which is of course independent of the system of coordinates chosen, can therefore in general be defined by

$$-p = \frac{1}{3} (T_{xx} + T_{yy} + T_{zz}) \quad (1.10)$$

3. The Tensorial Character of Electrodynanic Radiation

Pressure

In considering electrodynanic radiation pressure, we found that the force P_x due to a wave propagated in the x -direction is also directed in the x -direction. No force is acting in any direction perpendicular to the x -axis. So the radiation pressure is not a "pressure" in the sense in which we use this notion in hydrodynamics. Indeed, if we change the direction of our receiving surface in such a way that its outer normal makes an

angle ϑ with the positive x-axis (Fig. 4),

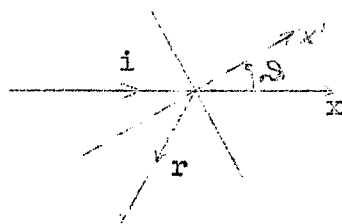


Fig. 4

Oblique incidence

the radiation force perpendicular to a perfectly reflecting surface is known to be

$$P_{x'} = 2\bar{E} \cos^2 \vartheta \quad (1.11)$$

It is by no means independent of the orientation of the surface with respect to the incident wave. This means that, strictly, one should speak of the radiation tensor, which at normal incidence has only one component, $T_{xx} = -P_x$. As only energy-densities are involved in $p = \bar{E}$, the expression for T is independent of the polarization of the plane electromagnetic wave:

$$T = \begin{vmatrix} -P_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad (1.12)$$

According to the rules for the transformation of a tensor T to a new set of axes, whose x, y -coordinates are rotated through an angle ϑ about the z -axis (Fig. 5), the tensor

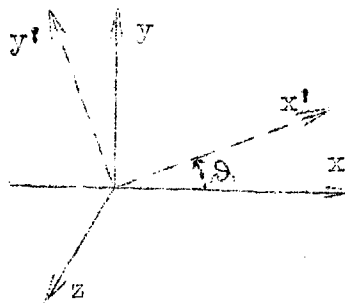


Fig. 5

Coordinate system x' , y' , z at oblique incidence

$$T = \begin{vmatrix} T_{xx} & 0 & 0 \\ 0 & T_{yy} & 0 \\ 0 & 0 & T_{zz} \end{vmatrix}$$

transforms into

$$T' = \begin{vmatrix} T_{xx} \cos^2 \alpha + T_{yy} \sin^2 \alpha & T_{x'y'} & 0 \\ T_{x'y'} & T_{xx} \sin^2 \alpha + T_{yy} \cos^2 \alpha & 0 \\ 0 & 0 & T_{zz} \end{vmatrix} \quad (1.13)$$

with $T_{x'y'} = \frac{1}{2} (T_{yy} - T_{xx}) \sin 2\alpha$.

As we have stated above, we find that the mean pressure

$$-p = \frac{T_{xx} + T_{yy} + T_{zz}}{3} = \frac{T_{x'x'} + T_{y'y'} + T_{z'z'}}{3}$$

remains unchanged; in liquids $T_{xx} = T_{yy} = T_{zz} = T = -p$, and using

Eq. (1.13) we find

$$T'_{liq} = \begin{vmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{vmatrix} = T_{liq}$$

But according to Eq. (1.13), the radiation tensor (1.12) transforms thus:

$$T'_{\text{rad}} = \begin{vmatrix} -P_x \cos^2 2\ell & \frac{P_x}{2} \sin 2\ell & 0 \\ \frac{P_x}{2} \sin 2\ell & -P_x \sin^2 2\ell & 0 \\ 0 & 0 & 0 \end{vmatrix} \quad (1.14)$$

We have discussed the difference between the concept of a hydrodynamic pressure, which is the same in all directions, and the physical properties of the electrodynamic radiation pressure, in some detail, because it will turn out that also in the case of acoustic radiation pressure the tensorial character of this quantity must be taken into account.

The electromagnetic radiation pressure is proportional to the mean total energy density and therefore to the square of the amplitudes of the electric and magnetic field-strengths (Eq. 1.3). This holds for all field-strengths, whether small or large. As to the acoustic radiation pressure, we shall see that a similar law is valid only for sufficiently small amplitudes of vibration.

4. Radiation Pressure in Acoustics

Rayleigh was the first to apply the concept of radiation pressure to mechanical waves in gases.¹³ He established relations for the average value in time of the pressure produced by an infinite plane wave in fluids and showed that this mean pressure is proportional to the mean energy density of the wave motion. The factor of proportionality was found by him not to be one in general, as in the case of electromagnetic waves, but dependent on the

special law connecting the pressure p and the density ρ in the fluid under consideration. The physical quantity called by Rayleigh the "pressure of vibrations", is, however, not identical with what is measured usually as "radiation pressure". The "radiation pressure" of a compressional wave striking a perfect absorber equals exactly twice the mean kinetic energy density of the wave motion, as we shall see. For small amplitudes this expression becomes identical with the mean total energy density and we obtain in this case the same relation as in the electromagnetic case.

The physical picture of radiation pressure is less simple in acoustics than in electrodynamics, where we have to do with the linear set of Maxwell's equations and with transverse waves. In the latter case the radiation pressure is determined by the electric and magnetic field-strengths and the formula for the radiation pressure is valid for all field intensities.

On the other hand, the equations describing the motion of acoustic waves are in general not linear. To avoid mathematical difficulties, the equations are usually "linearized" by using developments of the non-linear expressions in Taylor series and retaining only first-order terms. The characteristic quantity of these developments is the particle-displacement. The results of this "linearized" theory are therefore first approximations which can be expected to be valid only for small displacements. Radiation pressure, however, is connected with energy-densities, which are quadratic terms, containing the squares of displacements or velocities. Any theory dealing with radiation pressure must therefore retain at least all second order terms to avoid erroneous results. Even if this is done, the results are still only valid for small

amplitudes, if Taylor expansions are used. Simple relations between radiation pressure and mean total energy density (as for instance $\bar{P}_{\text{rad}} = \bar{E}$) are found only for small amplitudes of displacement. Relations of this kind do not express a basic law, as they do in electrodynamics, where they are independent of the amplitudes of the fields.

For the special case of a liquid we shall show that for large amplitudes no simple relation between radiation pressure and mean total energy density exists. Only in the second-order approximation of the general formula do we find proportionality between these quantities.

Each mass particle in a simple-harmonic acoustic compressional wave makes sinusoidal movements around the point it would occupy if there were no wave motion. The customary, and mathematically the most convenient, way to describe the wave propagation makes use of this displacement of any particle from its original location. All other physical quantities, as velocities or pressures, are related to the moving particles. The equation of motion that relates always to the same moving particle is usually called the "Lagrangian equation of motion", or the "equation of motion in Lagrangian coordinates".

The other way of dealing with these problems uses an equation of motion that relates to fixed coordinates in space, called "the equation of motion in Eulerian coordinates". Here we are not following the motion of a certain particle, but are observing what happens at a fixed point.

A careful distinction must be made between velocities, densities, pressures, and other quantities related to the instantaneous

position of a moving particle, and the same quantities when related to a fixed point in space. In dealing with the mean radiation pressure upon a reflector, for instance, we shall assume the average position of the reflector to be stationary in space. But we may also speak of the mean pressure observed at a surface which is moving together with the particles of a plane wave.

An acoustic compressional harmonic wave causes two important effects: 1) It changes the mean hydrostatic pressure at all points affected by the wave. 2) It causes additional mean stresses in the medium due to the time average of the periodic flow of mechanical momentum of the compressional wave.

If a plane acoustic wave of infinite width traverses a medium and strikes a perfect absorber, the mean pressure inside the wave region undergoes some diminution due to the wave motion. But if the acoustic beam is surrounded by undisturbed regions of the medium, as is ordinarily the case, the mean pressure becomes equalized throughout the entire fluid. In this case the effect 1) mentioned above does not cause a directed force upon obstacles placed in the path of the wave. The basic physical cause of the acoustic radiation pressure, or better radiation tensor, is in this case the time average of the periodic flow of mechanical momentum in the region affected by the beam. (See Part II, Sec. 12).

In a non-viscous fluid affected by plane acoustic waves, there is no net flow of matter due to the wave propagation. (The unidirectional "hydrodynamic flow" caused by acoustic waves is an additional physical effect due to the viscosity of the fluid and is not to be taken into consideration here; see also Part II, Sec. 10).

Considering a section of unit area of a plane compressional

wave (Fig. 6) which we imagine as fixed in space,

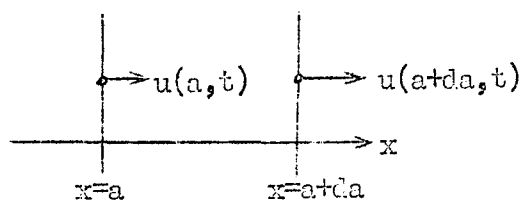


Fig. 6

Particle movement through
control-areas fixed in space

the average value in time of the mass flow is

$$\overline{\rho u} = \frac{1}{T_p} \int_t^{t+T_p} \rho(a,t) \cdot u(a,t) dt$$

$T_p = 1/f$ is the period, $\rho(a,t)$ the mass density at any moment t at the fixed coordinate $x = a$ of the cross-section under consideration, and $u(a,t)$ the instantaneous velocity of the particles crossing the section a at the time t . At different times different particles are crossing the section a , so that u and ρ refer only to the particles that cross at the time t . For a periodic wave motion in a medium that was originally at rest ($\bar{u}^* = 0$), $\overline{\rho u}$ vanishes at small amplitudes.

Next we consider the flow (or flux) of momentum through the section a . Arguing in the same way as in the preceding case of the electromagnetic flow of momentum, the momentum mu crossing unit area at $x = a$ in one second is $mu = (\rho u) \cdot u = \rho u^2$. This quantity is obviously always positive. This means that the plane wave carries periodically momentum - whose average value in time does not vanish - through fixed sections perpendicular to the direction of wave propagation.

The quantity $\rho u = (\rho u) \cdot u = \rho u^2$ can be interpreted in another way, leading to the concept of the "flux of momentum".

ρu is the density of mechanical momentum and u is the velocity with which the quantity ρu crosses a section. Thus we can say that the "flux of momentum-density" per second through a section of unit area equals $\rho u \cdot u = \rho u^2$, just as we speak of a "flux of mass-density", $\rho \cdot u$ per second over a unit area.

For small amplitudes the time average of the kinetic energy-density $1/2 \overline{\rho u^2}$ is equal to the time average of the potential energy. In this case $1/2 \overline{\rho u^2}$ is equal to $\bar{E}/2$ (\bar{E} = mean total energy density). The average value in time of the "flux of momentum density" $\overline{\rho u^2}$ is therefore equivalent to the mean total energy density \bar{E} .

Considering a volume element between two unit cross sections at a and $a + da$ (Fig. 6), a flux of momentum $\rho(a,t) \cdot u^2(a,t)$ enters the section a and a flux $\rho(a + da,t) \cdot u^2(a + da,t)$ leaves the section $(a + da)$ at the same time. The mean value $-\frac{\partial(\overline{\rho u^2})}{\partial a} \cdot da$ in time is obviously the gain in momentum of the volume element between a and $a + da$ per second. According to Newton's Law, this gain in momentum per second is equivalent to a force exerted upon the volume element. The volume element reacts with a force $+ \frac{\partial \overline{\rho u^2}}{\partial a} da$ to preserve equilibrium.

We can compare this picture with the volume element under the action of a stress T_{xx} in the x -direction (Fig. 7).

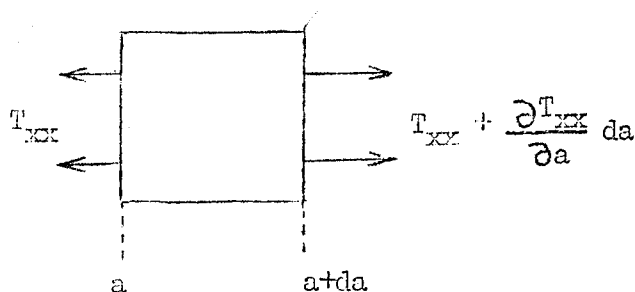


Fig. 7

Volume element under stress
in the x-direction

The force exerted upon the volume element is known to be $+\frac{\partial T_{xx}}{\partial a} da$.

The mean flux of momentum $\overline{\rho u^2}$ can therefore be considered with regard to its mechanical action as equivalent to a mean stress

$\bar{T}_{xx} = -\overline{\rho u^2}$, acting on any cross section affected by a compressional wave. The effective mean stress component \bar{T}_{xx} in the fluid due to the plane wave is therefore

$$\bar{T}_{xx \text{ eff.}} = \bar{T}_{xx} - \overline{\rho u^2} = -(\bar{p} + \overline{\rho u^2})$$

\bar{p} is the mean pressure and $\overline{\rho u^2}$ the mean flux of momentum density at a fixed section.

We have attempted to present a simple physical picture for the case of a plane wave that is in accord with a strict theoretical treatment of the motion of mechanical waves. In the general case of a three-dimensional motion, with velocities u_x , u_y , u_z , it can be shown that the resultant stress tensor in a medium in Eulerian coordinates becomes (see ref. 5, pp. 241 and 290):

$$T = \begin{vmatrix} T_{xx} - \rho u_x^2 & T_{xy} - \rho u_x u_y & T_{xz} - \rho u_x u_z \\ T_{xy} - \rho u_x u_y & T_{yy} - \rho u_y^2 & T_{yz} - \rho u_y u_z \\ T_{xz} - \rho u_x u_z & T_{yz} - \rho u_y u_z & T_{zz} - \rho u_z^2 \end{vmatrix} \quad (1.15)$$

The components of the flux of momentum in general are $(\rho u_i) \cdot u_k$ and are physically equivalent to the components of a stress-tensor.

The hydrostatic pressure of the fluid in absence of wave motion is denoted by p_0 , while p represents the change in p_0 due to the wave motion (excess pressure). The total effective pressure is

$p_t = p_0 + p$. For a plane wave in a fluid without absorption, $u = u_x$, $T_{xx} = T_{yy} = T_{zz} = -p_t$; Eq. (1.15) reduces, in time average, to

$$\bar{T} = \begin{vmatrix} -(\bar{p}_t + \overline{\rho u_x^2}) & 0 & 0 \\ 0 & -\bar{p}_t & 0 \\ 0 & 0 & -\bar{p}_t \end{vmatrix} \quad (1.16)$$

It is the term $(\bar{p} + \overline{\rho u_x^2})$ that turns out to be responsible for the radiation pressure exerted by a plane acoustic wave. According to Eq. (1.16), this quantity obviously has the character of a stress.

Considering a material surface element perpendicular to the direction of wave propagation x , and the adjacent volume element of a fluid (Fig. 8), the surface-element must exert a stress

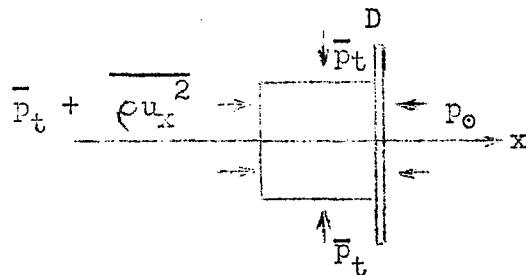


Fig. 8

Forces exerted upon a volume element adjacent to a device D for measuring radiation pressure.

- $(\bar{p}_t + \overline{\rho u_x^2})$ upon the right side of the volume element in order to maintain equilibrium. The surface therefore undergoes a pressure $\bar{p}_t + \overline{\rho u^2}$ in the x-direction on its left side. The static pressure behind the surface is denoted by p_0 . The resulting mean pressure exerted on the surface is therefore,

$$\bar{P}_{\text{rad}} = \bar{p}_t - \bar{p}_0 + \overline{\rho u_x^2} = \bar{p} + \overline{\rho u_x^2} \quad (1.17)$$

This is the general formula for the radiation pressure of a plane wave upon a plane material surface perpendicular to the direction of wave propagation. If the wave region communicates (for example by a small hole in the absorbing surface) with the medium behind the surface, $\bar{p}_t = \bar{p}_0$ at a perfect absorber and $\bar{P}_{\text{rad}} = \overline{\rho u_x^2}$. For small amplitudes, $\overline{\rho u_x^2} = \bar{E}$, and therefore, $\bar{P}_{\text{rad}} = \bar{E}$. (1.18)

For a perfectly reflecting surface the physical picture is more complicated, as we shall show later in the exact treatment; here \bar{p}_t at the surface differs from p_0 and turns out to be $p_0 + 2\bar{E}$ for small amplitudes, therefore $\bar{P}_{\text{rad}} = 2\bar{E}$ for such a surface perpendicular to the wave propagation.

Thus finally, we have arrived at the same relations as we found to hold in electrodynamics, but the physical background is more complicated in the case of mechanical waves. Other cases will be treated mathematically in this report. Our purpose here is to give a preliminary idea of the general causes leading to an acoustic radiation pressure, for some special and simple cases.

PART II

PLANE COMPRESSIONAL WAVES AND RADIATION PRESSURE IN LIQUIDS

1. General Considerations - Compressibility

As a whole, this report deals mainly with the radiation pressure of plane acoustic waves in liquids. There are two special reasons for this.

First, because this case is of practical importance. Although a plane wave cannot strictly be realized experimentally, still if the width of the acoustic beam is large in comparison with the wavelength, the concept of a plane wave gives us a good approximation, especially in the case of the high frequencies used in ultrasonic waves.

Second, because in the acoustics of liquids we are able to make use of the concept of constant compressibility. This concept introduces a simple analytical relation between the hydrodynamic pressure p and the relative change in volume $\Delta V/V_0$ of a volume element having the original volume V_0 , on which a pressure Δp is exerted. The compressibility β is by definition:

$$\beta = - \frac{\Delta V}{V_0} \cdot \frac{1}{\Delta p} \quad (2.1)$$

For small changes in volume and pressure, we have the isothermal compressibility,

$$\beta_{\text{isoth.}} = \frac{1}{V_0} \left(\frac{\partial V}{\partial p} \right)_T \quad (2.2)$$

and for the adiabatic compressibility

$$\beta_{\text{adiab.}} = \frac{1}{V_0} \left(\frac{\partial V}{\partial p} \right)_s \quad (2.3)$$

the subscripts T and σ denoting constant temperature and entropy.

For acoustic waves one uses normally the value $\beta_{\text{adiab.}}$, though the process of compression and rarefaction of the liquid is certainly not strictly adiabatic, owing to the unavoidable dissipation of heat. Still the exact value of β will not be very much affected even if the process is not strictly adiabatic. This fact can be seen from the formula giving the difference between $\beta_{\text{adiab.}}$ and $\beta_{\text{isoth.}}$ ⁴:

$$\beta_{\text{adiab.}} - \beta_{\text{isoth.}} = \frac{1}{V_0} \left[\left(\frac{\partial V}{\partial p} \right)_{\sigma} - \left(\frac{\partial V}{\partial p} \right)_T \right] = \frac{T}{c_p} \left(\frac{\partial V}{\partial T} \right)_p^2 \quad (2.4)$$

Numerically it turns out that this difference is rather small for liquids forming drops. It amounts to only a few percent under normal conditions of temperature and pressure. Thus for water, the value $\beta_{\text{adiab.}} = 46(10^{-6}) \text{ cm}^2/\text{kg force}$ at $t = 20^\circ\text{C}$ is ordinarily used for compressional waves.

Over a large range of pressure the compressibility is not constant. At $t = 20^\circ\text{C}$, for instance, $\beta_{\text{isoth.}}$ has the following mean values between $p = 1$ and $p = 1 + \Delta p$:

$\Delta p =$	0	100	500	1000	2000 kg/cm ²
$10^6 \frac{1}{V_0} \left(\frac{\partial V}{\partial p} \right)_T =$	46	46	43	40	35

The excess pressure in a plane compressional wave is a function of the intensity J of the wave. Its maximum value is given by

$$p_{\text{max}} = \sqrt{2J \rho c} \quad (2.5)$$

where c is the velocity of sound.

For water, at the very high intensity of $J = 100 \text{ watt/cm}^2$, p amounts approximately to 17 atm. Thus over the range usually encountered, the compressibility can be regarded as practically constant. This fact greatly simplifies the mathematical treatment of acoustic waves in liquids, and enables us also to extend the discussion to finite amplitudes in liquids.

2. Strains and Stresses in Viscous Liquids for Plane Compressional Waves

The general relations between stresses and strains in a viscous medium having a coefficient of viscosity η and a compressibility β are:

$$\begin{aligned} T_{xx} &= \frac{\Delta}{\beta} + \frac{2}{3} \eta (2 \epsilon_x - \epsilon_y - \epsilon_z) \\ T_{yy} &= \frac{\Delta}{\beta} + \frac{2}{3} \eta (2 \epsilon_y - \epsilon_z - \epsilon_x) \\ T_{zz} &= \frac{\Delta}{\beta} + \frac{2}{3} \eta (2 \epsilon_z - \epsilon_x - \epsilon_y) \\ T_{xy} &= 2\eta \epsilon_{xy}, \quad T_{yz} = 2\eta \epsilon_{yz}, \quad T_{zx} = 2\eta \epsilon_{zx} \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} \Delta &= \epsilon_x + \epsilon_y + \epsilon_z \\ \epsilon_x &= \frac{\partial \xi}{\partial x} \quad \epsilon_y = \frac{\partial \eta}{\partial y} \quad \epsilon_z = \frac{\partial \zeta}{\partial z} \\ \epsilon_{xy} &= \frac{1}{2} \left(\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial x} \right), \text{ etc.} \end{aligned} \quad (2.7)$$

ξ, η, ζ = displacements in the x, y, z directions

A dot denotes the partial time-derivative. For example,

$$\dot{\epsilon}_x = \frac{\partial}{\partial t} \epsilon_x = \frac{\partial}{\partial t} \frac{\partial \xi}{\partial x} = \frac{\partial^2 \xi}{\partial x \partial t}$$

For the sake of generality we include here viscous forces, though later on we shall neglect viscosity. The concept of plane waves means that at all points of any plane perpendicular to the direction x of the wave-propagation the state of motion is the same and therefore $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$, also the displacement $\eta = \zeta = 0$.

Introducing these assumptions in Eq. (2.6) we find:

$$\begin{aligned} T_{xx} &= \frac{1}{\beta} \frac{\partial \xi}{\partial x} + \frac{4}{3} \eta \frac{\partial^2 \xi}{\partial x \partial t} \\ T_{yy} &= \frac{1}{\beta} \frac{\partial \xi}{\partial x} - \frac{2}{3} \eta \frac{\partial^2 \xi}{\partial x \partial t} \\ T_{zz} &= \frac{1}{\beta} \frac{\partial \xi}{\partial x} - \frac{2}{3} \eta \frac{\partial^2 \xi}{\partial x \partial t} \\ T_{xy} &= T_{yz} = T_{zx} = 0. \end{aligned} \quad (2.8)$$

The pressure p is defined by:

$$-p = \frac{T_{xx} + T_{yy} + T_{zz}}{3} = \frac{\Delta}{\beta} \quad (2.9)$$

The quantity $T_{xx} + T_{yy} + T_{zz}$ is known to be an invariant of the tensor T_{ik} and independent of the orientation of the axes of the coordinate system. This is the reason for the definition (2.9), which is also in agreement with our earlier definition of compressibility in Eq. (2.1), because, as is well known, the quantity $\Delta = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$ is identical with the relative change in volume $\Delta V/V_0$ of a volume element V_0 . In the case of a plane wave one finds from Eqs. (2.9) and (2.8):

$$-p = \frac{1}{\beta} \frac{\partial \xi}{\partial x} \quad (2.10)$$

The terms containing η cancel out, as we expect, because p in (2.10) gives the mean pressure in a volume element due to the wave motion, which is independent of viscous forces and the same in all directions. In a plane wave a volume element is, it is true, stretched and compressed only in the direction x of the wave propagation, because the displacements have only one component ξ . But according to Eqs. (2.1) and (2.10) this causes a hydrodynamic pressure p , which is a scalar and therefore independent of direction. The relation (2.10) is exact by definition for media with constant compressibility; no higher order terms in ξ need be considered.

3. The Eulerian Equations of Motion for Plane Compressional Waves in Viscous Liquids

For treating the radiation pressure in a plane compressional wave we must know the stresses in the medium. As the radiation pressure acts on the surfaces of bodies inserted in the medium, whose mean position in space can be regarded as fixed, we desire to know the stresses in a coordinate system referring to points fixed in space. The hydrodynamic equations applying to this case are the so-called "Eulerian" equations. All physical quantities, as the particle velocity u or the pressure p , are regarded as functions of the coordinates x, y, z , of an axial system fixed in space, and of the time t . For a plane compressional wave with the only component of displacement ξ in the x -direction, the Eulerian equation of motion in the direction x of wave-propagation is known to be

$$\rho \frac{Du}{Dt} = \frac{\partial T_{xx}}{\partial x} \quad (2.11)$$

where ρ is the mass-density, Du/Dt the total derivative with respect to t of the velocity u in the x -direction of a volume-element having the density ρ . For the plane wave here considered, $u = u(x, t)$ is a function of x and t alone; for simplicity we use u instead of u_x . Eq. (2.11) follows immediately from consideration of the acceleration which a volume-element undergoes under the action of the force $\frac{\partial T_{xx}}{\partial x}$ in the x -direction. No accelerating forces exist in the y and z directions, because

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0 \text{ and } T_{ik} = 0 \text{ for } i \neq k$$

From (2.11) we have

$$\rho \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} \right) = \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} = \frac{\partial T_{xx}}{\partial x} \quad (2.12)$$

The conservation of mass of a volume element limited by surfaces fixed in space (which we will call "control-surfaces") requires in the one-dimensional case of a plane wave

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0 \quad (2.13)$$

the well-known "equation of continuity".

Combining Eqs. (2.12) and (2.13) by adding $u \frac{\partial \rho}{\partial t}$ to (2.12), and using $\frac{\partial \rho}{\partial t}$ from (2.13), one obtains another known form of Euler's equation:

$$\left. \begin{aligned} \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} - \frac{\partial T_{xx}}{\partial x} &= 0 \\ \text{or } \frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial p}{\partial x} &= 0 \end{aligned} \right\} \quad (2.14)$$

This is the main equation upon which we base our further consideration of the stresses in a plane wave. As has been stated, the

quantities ρ , u , and T_{xx} are functions of x and t , where x is a coordinate fixed in space.

We now consider a volume element limited by two planes at x and $x + dx$ and with unit area perpendicular to the x -axis. ρ , u and T_{xx} are taken with respect to the fixed planes x and $x + dx$ respectively. When particles move with velocity u across these planes, $u(x, t)$ belongs to different particles at different instants; that is, always to the particular particle which just crosses the stationary plane x (or $x + dx$) at the time t . ρ is the density of the volume element between the "control-surfaces" of unit area at x and $x + dx$.

We can interpret the terms on the left side of Eq. (2.14) as three forces acting on the volume element between the control-surfaces, whose sum equals zero according to d'Alembert's principle.

The first term $\frac{\partial(\rho u)}{\partial t}$ gives the change in mechanical momentum ρu at x with respect to the time t . This can be interpreted as equivalent to a force of inertia exerted by the volume-element, as is usually done in mechanics.

The term $\frac{\partial T_{xx}}{\partial x}$ means the mechanical force acting on the volume element. According to Eq. (2.8) it is given by

$$\frac{\partial T_{xx}}{\partial x} = \frac{1}{\rho} \frac{\partial^2 \xi}{\partial x^2} + \frac{4}{3} \eta \frac{\partial^3 \xi}{\partial x^2 \partial t} \quad (2.15)$$

The third term $\frac{\partial(\rho u^2)}{\partial x}$ is due to the gain in momentum in unit time which the volume element undergoes, if a greater amount of momentum ρu enters the area at x than leaves the area at $x + dx$ per second. Paraphrasing L. Brillouin's terminology we call the

quantity ρu^2 "the flux of mechanical momentum density" ("Flux de quantité de mouvement"; see ref. 5, pp. 241 and 290). It plays a dominant part in dealing with radiation pressure.

The incoming flux of momentum in unit time at x is obviously $(\rho u \cdot u)_x$; the outgoing flux at $x + dx$ in the same time is $(\rho u \cdot u)_{x+dx}$. The difference $(\rho u \cdot u)_{x+dx} - (\rho u \cdot u)_x$ is given by

$$\frac{\partial(\rho u^2)}{\partial x} \cdot dx$$

This quantity is a contribution to the momentum of the element in just as true a sense as the term $\frac{\partial(\rho u^2)}{\partial t} \cdot dx$ and can be regarded therefore in the same way as a force acting on the element.

It is obviously convenient to add this force due to the flux of momentum ρu^2 at x to the stress-tensor T_{xx} . So we write (2.14) thus:

$$\frac{\partial \rho u}{\partial t} - \frac{\partial}{\partial x} (T_{xx} - \rho u^2) = 0 \quad (2.16)$$

The flux of momentum ρu^2 is equivalent to an additional stress-component, acting on each volume element of a medium, the particles of which are in motion. This stress reduces to a particularly simple term in the case of plane waves. For the general case of a movement of particles in different directions with velocities u_x, u_y, u_z in the x, y and z -directions, the stress-tensor T is to be completed in the way we have already mentioned in (1.15).

The complete dynamic stress-tensor in Eulerian coordinates for a plane wave is given by

$$T = \begin{vmatrix} T_{xx} - \rho u^2 & T_{xy} & 0 \\ T_{xy} & T_{yy} & 0 \\ 0 & 0 & T_{zz} \end{vmatrix} \quad (2.17)$$

with the values T_{ik} as given in Eq. (2.8).

Disregarding viscous forces, we find by introducing the pressure p due to the wave motion, according to Eq. (2.10), and adding the static pressure p_0 to p :

$$T = \begin{vmatrix} -(p_t + \rho u^2) & 0 & 0 \\ 0 & -p_t & 0 \\ 0 & 0 & -p_t \end{vmatrix} \quad (2.18)$$

as already given in (1.16).

Eq. (2.17) or (2.18) will answer all questions concerning the radiation pressure, because it gives the stress-components in a plane compressional wave. We see that the medium undergoes a non-isotropic state of tension due to the unidirectional flux of momentum in the x -direction.

We seek the solution of Eq. (2.16) for the case of pure sinusoidal waves, in order to insert the values of p_t and ρu^2 in Eq. (2.18).

The Eulerian equation (2.16) or (2.12) in fixed coordinates is nonlinear, because of terms like $u \frac{\partial u}{\partial x}$ or like $\frac{\partial (\rho u^2)}{\partial x}$.

This difficulty, which complicates the solution, can - at least in liquids - be avoided by using another set of equations of motion, which are usually called the "Lagrangian equations of motion". These equations relate not to points fixed in space, but rather to the moving particles. It is much easier to find the correct solution by means of the Lagrangian equations; for this reason they are usually

employed to find the solution for plane compressional waves. Nevertheless, for expressing the radiation pressure exerted on obstacles whose mean position in space can be regarded as fixed, these solutions must be transformed from the system related to moving particles into the Eulerian coordinate system. This can easily be done, as will be seen.

One important point, which must not be overlooked, is the circumstance that the average values in time of both terms p and ρu^2 in fixed coordinates are second-order terms. The radiation pressure is, as has been seen, proportional to the energy-density, and is therefore a second-order quantity; this explains also its relatively small numerical values in comparison with first-order pressures, even at the small amplitudes ordinarily used. Whereas the first-order pressure has maximal values up to some kilograms per cm^2 , the acoustic radiation pressure only reaches values of the order of grams or dynes of force per cm^2 .

In transforming p and ρu^2 from the Lagrangian into the Eulerian system all second-order terms must be carefully taken into account. Restriction to first-order solutions leads to erroneous results. Instances might be cited in certain papers dealing with radiation pressure.

In the case of a liquid the use of the Lagrangian equations gives us an exact solution for finite amplitudes, which can be transformed into the Eulerian coordinate system. In the case of gases, on the other hand, even the Lagrangian equation becomes nonlinear and we must develop the Lagrangian solution in series at least up to the second-order terms in order to find the radiation pressure for small amplitudes.

4. The Lagrangian Equation of Motion for Plane Compressional Waves in Viscous Liquids

Let ξ be the displacement of a particle with respect to its original undisturbed position x . If there is a wave motion, particles move back and forth through x , so that $\xi = \xi(x, t)$. The true particle coordinate in space at any time t is therefore $x + \xi(x, t)$. The equation of motion for the displacements ξ can easily be established and is found to be^{9, 12}

$$\rho_0 \frac{\partial^2 \xi(x, t)}{\partial t^2} = \frac{\partial T_{xx}^*}{\partial x} \quad (2.19)$$

This equation is absolutely exact, with nothing neglected. The constant ρ_0 is the original undisturbed density in absence of wave motion. $T^*(x, t)$ is the stress at the location of the moving particle, that is, at $x + \xi(x, t)$. We use the notation T_{xx}^* for stresses associated with moving volume elements for distinction from T_{xx} in Eulerian coordinates related to points fixed in space. The particle velocity at the instantaneous location $x + \xi(x, t)$ is denoted similarly by $u^*(x, t) = \frac{\partial \xi}{\partial t}$.

The relation between the quantities T_{xx} , u , and ρ in Eulerian fixed coordinates and the same quantities T_{xx}^* , u^* , and ρ^* in Lagrangian coordinates is obviously (see also ref. 16) :

$$\begin{aligned} T^*(x, t) &= T(x + \xi, t) \\ u^*(x, t) &= u(x + \xi, t) \\ \rho^*(x, t) &= \rho(x + \xi, t) \end{aligned} \quad (2.20)$$

or, by substituting $x - \xi$ for x ,

$$\begin{aligned} T^* (x - \xi, t) &= T (x, t) \\ u^* (x - \xi, t) &= u (x, t) \\ \rho^* (x - \xi, t) &= \rho (x, t) \end{aligned} \quad (2.21)$$

Eq. (2.20) or (2.21) allows us to find T , u , and ρ , for example, if we know the solution for T^* , u^* , ρ^* in Lagrangian coordinates.

For the special case of liquids with constant compressibility β the relation between T^* and ξ is given by Eq. (2.8), and from (2.19) we have the well-known equation of motion for viscous liquids:

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = \frac{1}{\beta} \frac{\partial^2 \xi}{\partial x^2} + \frac{4}{3} \eta \frac{\partial^3 \xi}{\partial x^2 \partial t} \quad (2.22)$$

This equation is linear and can therefore be solved easily and exactly, whereas the corresponding equation for the same case in Eulerian coordinates is nonlinear, as already mentioned. For gases the connection between T^* or p^* and the displacement in general is nonlinear. In this case, the solution of (2.19) can only be expressed in the form of a series.

5. The Exact Solution for Plane Compressional Waves in Non-viscous Liquids

Since we shall consider radiation pressure here without regarding the influence of viscous absorption, we neglect the last term on the right side of Eq. (2.22). The solution for plane waves in liquids follows at once from this equation. For pure sinusoidal motion of a particle about its original location the solution is known to be

$$\xi(x, t) = \xi_0 \left\{ \frac{\sin}{\cos} (\omega t - kx) + \gamma \frac{\sin}{\cos} (\omega t + kx + \theta) \right\} \quad (2.23)$$

with

$$k = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} = \frac{\omega}{c}; \quad \omega = 2\pi f;$$

$$c = \sqrt{\frac{1}{\beta \rho_0}} = \frac{\omega}{k}$$

This solution corresponds to the assumption, commonly made with respect to the boundary conditions, that the wave motion is generated by a piston-like source moving harmonically with angular frequency ω around its average position. The source may be located at any x or at infinity. From (2.23) we derive the following quantities:

$$u^*(x, t) = \frac{\partial \xi}{\partial t} = \omega \xi_0 \left\{ + \cos (\omega t - kx) + \gamma \frac{\cos}{-\sin} (\omega t + kx + \theta) \right\} \quad (2.24)$$

$$-p^*(x, t) = T^*_{xx} = \frac{1}{\beta} \frac{\partial \xi}{\partial x} = \frac{k \xi_0}{\beta} \left\{ -\cos (\omega t - kx) + \gamma \frac{\cos}{-\sin} (\omega t + kx + \theta) \right\} \quad (2.25)$$

The density ρ^* follows from

$$\rho^* V^* = \rho_0 V_0 \quad (2.26)$$

V_0 is the original volume of a volume element, V^* the volume of a displaced volume element. According to Eqs. (2.1) and (2.10),

$$V^* = V_0 + \Delta V = V_0 - \beta V_0 \Delta p_t^*$$

or

$$V^* = V_0 \left(1 + \frac{\partial \xi}{\partial x} \right) \quad (2.27)$$

From (2.26) we derive the density ρ^* at the coordinate of the moving particle:

$$\rho^*(x, t) = \frac{\rho_0}{1 + \frac{\partial \xi(x, t)}{\partial x}} \quad (2.28)$$

The solution given by Eqs. (2.23, 2.24, 2.25) are strictly exact, so long as the assumption of a constant compressibility can be regarded as valid.

However, there is one important restriction which limits these solutions to a definite range of amplitudes ξ_0 . The differential equation (2.22) implies the supposition that $\xi(x, t)$ and the derivatives $u^* = \partial \xi / \partial t$ and $-\rho p^* = \partial \xi / \partial x$ are unique functions of x and t . At a coordinate $x_1 + \xi$ in space, for instance, only one kind of particle with displacement $\xi(x_1)$ and velocity $u^*(x_1)$ at a certain time t is assumed to exist in the derivation which leads to Eq. (2.22). At the moment when a particle A, originally located at a position x_A behind another particle B originally at x_B , undergoes such a large displacement ξ_A that it reaches or passes the particle B at $x_B + \xi(x_B)$, we would find two different particles with, in general, two different velocities $u^*(x_A)$ or $u^*(x_B)$ at the same point $x_A + \xi(x_A) = x_B + \xi(x_B)$ in space at the same time. The condition to be imposed upon our solution (2.23) for ξ for preventing the overtaking of one particle by another is found to be

$$\frac{\partial \xi}{\partial x} > -1$$

This can be understood immediately from the expression for ρ^* in Eq. (2.28). If two different particles originally located at different coordinates x_A and x_B came into contact, the original volume between the planes x_A and x_B would be compressed to zero, and the density ρ^* would become infinite. This would happen, as may be seen from Eq. (2.28), if the denominator vanished or if $\partial \xi / \partial x = -1$. Only so long as the condition above is valid, does ρ^* remain finite; from this fact, together with Eq. (2.23) we find

$$\frac{\partial \xi}{\partial x} = k \xi_0 \left\{ \begin{array}{l} -\cos \\ +\sin \end{array} (\omega t - kx) + \gamma \begin{array}{l} \cos \\ -\sin \end{array} (\omega t + k_x + \theta) \right\} > -1$$

and hence the condition limiting the maximal amplitude ξ_0 of any displacement is

$$|k\xi_0| < 1 \quad \text{or} \quad |\xi_0| < \frac{1}{k} = \frac{\lambda}{2\pi} \quad (2.29)$$

For all amplitudes smaller than that indicated by (2.29), the solution (2.23) is exact and unique. In media with constant compressibility, compressional sinusoidal waves as represented by Eq. (2.23) are propagated without distortion. This fact has in principle already been stated by Rayleigh,¹³ though he did not deal specifically with real liquids, but made only a theoretical statement. The condition (2.29) is more a theoretical than a practical limitation for the amplitudes ξ_0 , as amplitudes in the neighbourhood of $k\xi_0 = 1$ would require enormous energies that could not be realized experimentally. (See Sec. 8 b)

By means of the transformation formulas (2.21) we are able now to find the exact solution for plane compressional waves in liquids in Eulerian coordinates. We need only replace x in Eq. (2.23) etc. by $x - \xi(x, t)$. Choosing the sine solution for ξ in (2.23), we obtain from (2.24) and (2.25)

$$\begin{aligned} u(x, t) &= \omega\xi_0 \left\{ \cos[\omega t - k(x - \xi)] + \gamma \cos[\omega t + \theta + k(x - \xi)] \right\} \\ &= \omega\xi_0 \left\{ \cos[\omega t - kx + k\xi_0 (\sin(\omega t - kx) + \gamma \sin(\omega t + \theta + kx))] \right. \\ &\quad \left. + \gamma \cos[\omega t + \theta + kx - k\xi_0 (\sin(\omega t - kx) + \gamma \sin(\omega t + \theta + kx))] \right\} \end{aligned} \quad (2.30)$$

and

$$\begin{aligned} p(x, t) &= \frac{k\xi_0}{\beta} \left\{ \cos[\omega t - kx + k\xi_0 (\sin(\omega t - kx) + \gamma \sin(\omega t + \theta + kx))] \right. \\ &\quad \left. - \gamma \cos[\omega t + \theta + kx - k\xi_0 (\sin(\omega t - kx) + \gamma \sin(\omega t + \theta + kx))] \right\} \end{aligned} \quad (2.31)$$

These expressions (2.30) and (2.31) are exact solutions of the nonlinear Eulerian differential equation of motion (2.14), for amplitudes $\xi_0 < \lambda/2\pi$ according to (2.29) and for the special case of pure sinusoidal waves traveling to the right and left given by the Lagrangian solution (2.23) for the displacement ξ . This statement can be verified by inserting Eq. (2.30) and the appropriate expression for ρ in the differential equation (2.14). (See Part II, Sec. 9 for a general proof).

The independent variables x and t enter this solution throughout in the combination $\omega t \pm kx$. Thus the solution in Eulerian coordinates also represents systems of waves traveling with the phase-velocity $c = \omega/k$ to the right and left. But the resultant wave motion is not given by simple superposition, such as holds with the Lagrangian coordinates. Physically this means that $u(x,t)$ and $p(x,t)$ in Eulerian coordinates are not represented in our case by superposition of pure sinusoidal traveling waves; the wave form is distorted and this fact is responsible for the existence of higher order terms in ξ_0 , whose average values in time do not vanish.

6. Development in Series up to the Second Order of the Exact Solution in Eulerian Coordinates

For small amplitudes ξ_0 we develop the solutions (2.30) and (2.31) in Taylor series, retaining second-order terms in ξ_0 . For $u(x,t)$ we get from (2.30), writing $\tau_{\mp} = \omega t \mp kx$:

$$u = \omega \xi_0 \left\{ \cos \left[\tau_- + k\xi_0 \left(\sin \tau_- + \gamma \sin(\tau_+ + \theta) \right) \right] + \gamma \cos \left[\tau_+ + \theta - k\xi_0 \left(\sin \tau_- + \gamma \sin(\tau_+ + \theta) \right) \right] \right\}$$

or,

$$u = \omega \xi_0 \left\{ \cos \tau_- - k \xi_0 \left(\sin \tau_- + \gamma \sin(\tau_+ + \theta) \right) \cdot \sin \tau_- + \gamma \cos(\tau_+ + \theta) + \gamma k \xi_0 \left(\sin \tau_- + \gamma \sin(\tau_+ + \theta) \right) \cdot \sin(\tau_+ + \theta) \right\} \quad (2.32)$$

As we shall be chiefly interested in average time values of u and p , we compute the mean value \bar{u} of u by

$$\bar{u}(x, t) = \frac{1}{T_p} \int_{t_1}^{t_1 + T_p} u(x, t) dt = \frac{1}{\omega T_p} \int_{\omega t_1}^{\omega t_1 + \omega T_p} u(x, \omega t) d\omega t$$

or,

$$\bar{u}(x, t) = \frac{1}{2\pi} \int_{t_1}^{t_1 + 2\pi} u(x, \omega t) d\omega t \quad (2.33)$$

From (2.32) and (2.33) we find

$$\bar{u} = -\frac{\omega k \xi_0^2}{2} (1 + \gamma \cos(2 kx + \theta)) + \frac{\omega k \gamma \xi_0^2}{2} (\gamma + \cos(2 kx + \theta))$$

or

$$\bar{u} = -\frac{\omega k \xi_0^2}{2} (1 - \gamma^2) \quad (2.34)$$

The result (2.34) shows that $\bar{u}(x, t)$ does not depend upon x even if there is a reflected wave, the latter being characterized by the reflection coefficient γ . In Lagrangian coordinates the time average \bar{u}^* of the velocity u^* equals zero.

In the same way as in deriving \bar{u} , we find from Eq. (2.31) for the average time value of p ,

$$\bar{p} = -\frac{k^2 \xi_0^2}{2\rho} (1 + 2\gamma \cos(2 kx + \theta) + \gamma^2) \quad (2.35)$$

The mean pressure \bar{p} in Eulerian coordinates evidently varies sinusoidally with x when reflected waves are present.

Another mean value that we shall need is the time-average value of the quantity ρu^2 in Eulerian coordinates. If we transform ρ^* into ρ and develop ρ in powers of ξ_0 , we obtain

$$\rho = \rho_0 (1 + a_1 \xi_0 + a_2 \xi_0^2 + \dots)$$

The development of u may be written

$$u = b_1 \xi_0 + b_2 \xi_0^2 + \dots = \xi_0 (b_1 + b_2 \xi_0 + \dots) \quad (2.36)$$

as seen from (2.32). So we find

$$\rho u^2 = \rho_0 (1 + a_1 \xi_0 + a_2 \xi_0^2) \cdot (b_1 + b_2 \xi_0)^2 \xi_0^2 + \dots$$

or, disregarding terms of higher than the second order,

$$\rho u^2 = \rho_0 b_1^2 \xi_0^2 + \dots = \rho^* u^{*2} \quad (2.37)$$

Therefore we do not need the development of ρ in powers of ξ_0 , as only the undisturbed density ρ_0 enters (2.37). This relation shows also that the value of the kinetic energy density at small amplitudes is the same in both Eulerian and Lagrangian coordinates.

We have from Eqs. (2.36), (2.37) and (2.32),

$$\overline{\rho u^2} = \rho_0 \xi_0^2 \cdot \overline{b_1^2} = \rho_0 \xi_0^2 \cdot (\omega \cos \tau_- + \omega \gamma \cos(\tau_+ + \theta))^2$$

or

$$\overline{\rho u^2} = \frac{\omega^2 \rho_0 \xi_0^2}{2} (1 + 2 \gamma \cos(2 kx + \theta) + \gamma^2) = \rho^* u^{*2} \quad (2.38)$$

Comparing \bar{p} and $\overline{\rho u^2}$ in (2.35) and (2.38), we find $\bar{p} = \overline{\rho u^2}$. This relation holds in liquids at small amplitudes, but

not in gases. (See Part III, Sec. 4).

It is conformable to our purpose to introduce the mean total energy density \bar{E} in (2.35) and (2.38). For a pure sinusoidal plane compressional wave traveling to infinity in one direction, we get the mean kinetic energy density $\frac{\rho u^2}{2}$ directly from (2.38), if we put $\gamma = 0$. Therefore for small amplitudes

$$\bar{E}_{\text{kin}} = \frac{\omega^2 \rho_0 \xi_0^2}{4} = \bar{E}_{\text{kin}}^* \quad (2.39)$$

The identity of \bar{E}_{kin} in Eulerian coordinates at small amplitudes with $\bar{E}_{\text{kin}}^* = \frac{\rho u^{*2}}{2}$ in Lagrangian coordinates has already been seen in Eq. (2.37).

The potential energy density due to the elastic compression of the medium is given by $dE_{\text{pot}} = -p \cdot dV$, that is the work done upon unit volume $V_0 = 1$ under the action of a pressure p . From Eq. (2.1) we find, with $V_0 = 1$, $dV = -\beta dp$ and therefore $dE_{\text{pot}} = \beta p dp$. By integration we have

$$E_{\text{pot}}^* = \frac{1}{2} \beta p^{*2}$$

Inserting p^* from (2.10) we obtain

$$E_{\text{pot}}^* = \frac{1}{2\beta} \left(\frac{\partial \xi}{\partial x} \right)^2$$

Since ξ , according to Eq. (2.23), is a function of the argument $\omega t + kx$ alone,

$$\frac{\partial \xi}{\partial x} = \mp \frac{k}{\omega} \frac{\partial \xi}{\partial t} = \mp \frac{1}{c} \frac{\partial \xi}{\partial t} = \mp \frac{u^*}{c}$$

So we have, remembering that $c^2 = 1/\beta \rho_0$,

$$E_{\text{pot}}^* = \frac{1}{2\beta} \frac{u^{*2}}{c^2} = \frac{1}{2} \rho_0 u^{*2} \quad (2.40)$$

The difference between E_{kin}^* and E_{pot}^* in Lagrangian coordinates is therefore $E_{\text{kin}}^* - E_{\text{pot}}^* = (\rho^* - \rho_0) u^{*2}/2$ in liquids. It vanishes at small amplitudes, that is in the second-order approximation.

At small amplitudes the average time value for the potential energy density in pure sinusoidal waves is given therefore by the same expression as for the mean kinetic energy density in (2.39):

$$\bar{E}_{\text{pot}}^* = \frac{\omega^2 \rho_0 \xi_0^2}{4} = \bar{E}_{\text{kin}}^* \quad (2.41)$$

The total mean energy density of a plane unidirectional wave at small amplitudes is

$$\bar{E}^* = \bar{E}_{\text{kin}}^* + \bar{E}_{\text{pot}}^* = \frac{\omega^2 \rho_0 \xi_0^2}{2} = \bar{E} \quad (2.42)$$

For small amplitudes there is therefore no distinction between \bar{E}^* and \bar{E} .

Introducing Eq. (2.42) in (2.38) and (2.35) and remembering that $k^2/\beta = \omega^2/\beta c^2 = \rho_0 \omega^2$, we find

$$\bar{p} = -\bar{E} (1 + 2 \gamma \cos(2 kx + \theta) + \gamma^2) \quad (2.43)$$

$$\overline{\rho u^2} = +\bar{E} (1 + 2 \gamma \cos(2 kx + \theta) + \gamma^2) \quad (2.44)$$

We have now derived the expressions in Eulerian coordinates needed to establish the stress tensor.

7. Stress-tensor and Radiation Pressure for Small

Amplitudes in Liquids

By inserting Eqs. (2.43) and (2.44) in (2.18), we obtain the time average of the stress tensor T in a liquid traversed by plane compressional waves with small amplitudes ($k\xi_0 \ll 1$), disregarding viscous absorption. As the liquid is under an additional hydrostatic pressure p_0 , we must add this pressure to the hydrodynamic pressure \bar{p} in (2.43) due to the wave motion alone. So we have for the mean stress tensor \bar{T}

$$\bar{T} = \begin{vmatrix} -p_0 & 0 & 0 \\ 0 & -(p_0 - \bar{E}[1 + 2\gamma \cos(2kx + \theta) + \gamma^2]) & 0 \\ 0 & 0 & -(p_0 - \bar{E}[1 + 2\gamma \cos(2kx + \theta) + \gamma^2]) \end{vmatrix} \quad (2.45)$$

Owing to the fact that the two quantities \bar{p} in (2.43) and $\overline{\rho u^2}$ in (2.44) are numerically equal but of opposite signs, the sum $\bar{p} + \overline{\rho u^2}$ equals zero. Thus T_{xx} in (2.45) is equal to the static pressure p_0 and is not changed by the wave motion. The existence of a compressional wave evidently changes only the components T_{yy} and T_{zz} perpendicular to the direction of wave propagation. From Eq. (2.43) it is seen that the presence of a compressional wave in a liquid diminishes the mean static pressure p_0 by the amount $\bar{E}(1 + 2\gamma \cos(2kx + \theta) + \gamma^2)$. In the direction x of wave propagation this diminution is exactly compensated by the stress $\overline{\rho u^2}$ due to the flux of momentum in the same direction.

Now let us insert an infinite plane material surface perpendicular to x . The compressional wave traveling to the right undergoes reflection at this surface, characterized by the

coefficient γ of the reflected amplitude and the phase angle θ of the reflected wave. For computing the force acting upon the reflector, we assume that behind the reflector is the same liquid under the static pressure p_0 . The resultant force per square centimeter acting upon the reflector, which is the radiation pressure, equals the difference between the pressures on the left and right sides of the reflector. (Fig. 9)

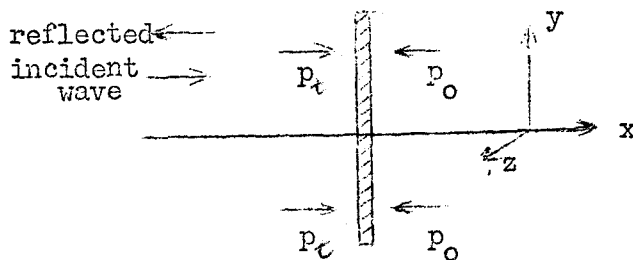


Fig. 9

Plane reflector undergoing a pressure p_t on the front side and a pressure p_0 on the rear

The pressure exerted at the boundary between the reflector and the irradiated medium is given by

$$X_n = T_{xx} \cos(n, x) + T_{xy} \cos(n, y) + T_{xz} \cos(n, z) \quad (2.46)$$

X_n denotes the component of pressure in the x -direction and n the direction of the inner normal to the surface. In our case (Fig. 9) only $\cos(n, x) = -1$ is different from zero, furthermore $T_{xy} = T_{xz} = 0$ according to (2.45). There remains only a pressure $X_n = -T_{xx} = p_0$ perpendicular to the reflector on its left side. As the same pressure p_0 exists at the right side of the reflector, no resultant force is exerted upon the reflecting surface. A plane compressional wave extending to infinity in all directions perpendicular to the wave propagation in a liquid

would exert no radiation pressure upon any plane infinite reflector perpendicular to the direction of the wave propagation.

This case of a plane wave extending to infinity is merely theoretical, as it can hardly, even approximately, be realized experimentally. The acoustic beam always has a finite cross-section and is usually surrounded by a part of the same liquid which is not affected by the wave motion. The mean pressure in the surrounding liquid is p_0 ; the pressure inside the beam is changed by the wave motion to $p_0 + \bar{p}$. As \bar{p} is negative according to Eq. (2.43), the mean pressure inside the beam is lower than in the surrounding medium.

The mean pressure tends to be equalized over the whole liquid. This means that the surrounding liquid, where the pressure p_0 is assumed to be maintained constant, compresses the beam-region until the mean pressure in the beam is the same as that around it. This effect leads to a radiation pressure upon a finite material obstacle placed in the way of the acoustic beam.

The simplest case, and that which has received most attention in the literature, is the radiation pressure upon a perfectly absorbing surface. There no reflected wave exists and $\gamma = 0$. The theoretical concept often used to represent a perfect absorber regards the absorbing wall as free to move in a direction normal to its plane and following the movement of the particles immediately adjacent to the wall. The pressure observed at the wall is identical then with the pressure p^* associated with

the moving particles, which we know to be purely sinusoidal from Eq. (2.25), whether the beam is finite or infinitely extended.

The mean work done by p^* per second and square centimeter at the wall equals the average value in time

$$\frac{1}{T_p} \int_0^{T_p} p^* d\zeta = \frac{1}{T_p} \int_0^{T_p} p^* \frac{\partial \zeta}{\partial t} dt = \frac{1}{T_p} \int_0^{T_p} p^* u^* dt,$$

which by use of Eqs. (2.24) and (2.25) with $\gamma = 0$, and from (2.42), is easily computed to be equal to $\bar{E} \cdot c$. Such a moving wall would absorb indeed all the energy reaching it, and there would be no reflection. The mean force $\frac{1}{T_p} \int_0^{T_p} p^* dt$ exerted upon such a moving wall would be zero, as p^* is purely sinusoidal. The freely moving and absorbing wall would be indeed a perfect absorber, but it represents more a theoretical fiction than a feasible experimental device.

In measuring radiation pressure we do not follow the movement of the particles of the surface struck by the acoustic beam. All particles on the surface move periodically around their original positions, so that the center of mass of the surface, when averaged over a whole period, can be regarded as fixed in space. Usually the surface subjected to radiation pressure is also connected to a measuring device of considerable inertia, unaffected by the rapid motion of the particles in the wave. This is the reason why we introduce the Eulerian system of coordinates fixed in space for computing the time-average of the radiation pressure related to a device which is assumed to be fixed in space rather than to follow the motion of particles. (See also Sec. 12).

As in optics, the most practicable approach to a perfect absorber is the "hohlraum" or radiation trap. An acoustic hohlraum consists of a cavity with acoustically insulating walls, filled with an absorbing medium, and provided with a small aperture through which the acoustic beam is admitted. Such a device, in the form of a cylindrical tube closed at one end, has been used for measuring acoustic intensities in water.* For frequencies in the megacycle range the absorption of energy is practically complete in a tube that is not excessively long. The plane of the aperture therefore serves as a totally absorbing surface.

We now consider first the radiation pressure exerted upon a perfectly absorbing surface, and then the somewhat more complicated case of a reflecting surface.

a. Radiation pressure upon a perfectly absorbing surface, disregarding viscosity.**

From Eq. (2.45) we find, for $\gamma = 0$, the average-time value of \bar{T} :

$$\bar{T}_{\gamma=0} = \begin{vmatrix} -p_0 & 0 & 0 \\ 0 & -(p_0 - \bar{E}) & 0 \\ 0 & 0 & -(p_0 - \bar{E}) \end{vmatrix} \quad (2.47)$$

As has already been pointed out, the mean pressure in the acoustic beam is lowered by the wave motion by an amount equal to the mean energy density \bar{E} ***. If the beam of infinite cross-section

* Contract N6 ONR-262, Task Order 1: Report No. 1, Dec. 20, 1947, p. 36; Report No. 3, Feb. 21, 1949, pp. 8-11.

** See also ref. 5, pp. 302-304

*** An instructive physical outline of this effect is given in a paper by Hertz and Mende⁸. (See also ref. 1)

does not communicate with the region of the liquid unaffected by the wave motion, the radiation pressure upon an infinite wall, one side of which is struck by the wave and whose other side is adjacent to the undisturbed liquid, is zero because both sides of the wall are subject to the same force p_0 per square centimeter in the x-direction.

The change in the time-average value of the density ρ^* due to the wave motion is found from Eq. (2.28). For small amplitudes, regarding p^* as a small quantity, we have $\rho^* = \rho_0 (1 + \beta p^* + \beta^2 p^{*2} + \dots)$. After transformation into Eulerian coordinates, we obtain the mean change in density at small amplitudes

$$\overline{\Delta \rho} = \bar{\rho} - \rho_0 = \rho_0 \beta (\bar{p} + \overline{\beta p^2}) \quad (2.48a)$$

As the pressure inside the beam is lowered to $p_0 - \bar{E}$, \bar{p} equals $-\bar{E}$. $\overline{\beta p^2}$ is found immediately from Eqs. (2.31) and (2.42) to be $+\bar{E}$ for $\gamma = 0$ and small amplitudes. Therefore we obtain from (2.48a) $\overline{\Delta \rho} = 0$, which means that the wave motion in the infinitely extended beam does not change the mean density in the second-order approximation, though it lowers the mean pressure by the amount \bar{E} .

If the beam is of finite cross-section and surrounded by liquid with hydrostatic pressure p_0 , the pressure inside the beam being $p_0 - \bar{E}$, the liquid in the beam undergoes a compression, until the mean pressure inside reaches the same value p_0 as in the surrounding region. The change in density $\overline{\Delta \rho'}$ due to this compression is found from $\Delta \rho' / \rho_0 = - \Delta V' / V_0 = \beta \Delta p$ or

$$\overline{\Delta \rho'} = \rho_0 \beta \overline{\Delta p} = \frac{\overline{\Delta p}}{c^2} = \frac{\bar{E}}{c^2} \quad (2.48b)$$

Thus the wave motion causes a slight increase in density $\overline{\Delta \rho'}$ inside a beam of finite width, though the pressure becomes the same in the beam as in the surrounding undisturbed medium with the unchanged density* ρ_0 .

This fact can perhaps best be understood by the following reasoning: The acoustic wave motion changes the pressure by a second-order term ($-\bar{E}$), whereas the change in density is small of higher order. The compression of the beam of finite width by the surrounding medium changes both pressure and density by second order terms (\bar{E} and \bar{E}/c^2 respectively). In the resulting pressure the two changes cancel each other to zero so that the final pressure becomes p_0 again; in the final density however the sum of the two changes results in the second order term $\overline{\Delta \rho'} = \bar{E}/c^2$.

The resultant stress tensor becomes now (see also ref. 5, p. 302)

$$\bar{T}_{\gamma = 0} = \begin{vmatrix} -(p_0 + \bar{E}) & 0 & 0 \\ 0 & -p_0 & 0 \\ 0 & 0 & -p_0 \end{vmatrix} \quad (2.49)$$

According to (2.46) a resultant force \bar{E} per square centimeter is now exerted upon a fixed, perfectly absorbing wall. The radiation pressure upon such a surface therefore equals the mean energy density, as long as the amplitude is small:

$$\underline{(\bar{P}_{\text{rad}})_{\gamma = 0} = \bar{E}} \quad (2.50)$$

* See also the paper by G. Richter¹⁵, who, following a different path, arrives at the same result concerning the mean density $\bar{\rho}$.

The radiation pressure in this case is due to the mean periodic flux of mechanical momentum due to the wave motion, which introduces the term $\overline{\rho u^2} = \bar{E}$ in \bar{T}_{xx} . It is independent of the direction of propagation of the wave (as is expected because it is a tensor-component and not a vector); this means that \bar{P}_{rad} has the same direction, whether an incident beam strikes the absorber or a reflected beam leaves it. The radiation pressure is not caused by the increase in density $\Delta \rho'$ inside the beam, due to compression of the surrounding liquid, as has often been asserted.^{8,1}

In the plane wave of infinite width, not in communication with undisturbed regions of the liquid, the term T_{xx} , responsible for the radiation pressure, was, according to (2.18), found to be $p_0 - \bar{E} + \overline{\rho u^2}$. The two terms $-\bar{E}$ and $\overline{\rho u^2} = +\bar{E}$ cancel exactly, which means that the diminution in mean pressure is just compensated by the flux of momentum. In the x-direction no additional stress acting on a volume element of the liquid would be found, if this were all that happened. By virtue of the interaction with the surrounding liquid the mean pressure $p_0 - \bar{E}$ is brought up to the value p_0 of the undisturbed liquid. The term $T_{xx} = p_0 - \bar{E} + \overline{\rho u^2}$ is changed into $p_0 + \overline{\rho u^2}$. Now an additional stress $\overline{\rho u^2}$ acts throughout the beam in the x-direction and leads to a radiation pressure $\bar{E} = \overline{\rho u^2}$ upon a perfectly absorbing wall perpendicular to the direction x of wave propagation.

b. Radiation pressure at a reflecting surface,
disregarding viscosity

In the case of a reflected wave the mean pressure over the beam is given by Eq. (2.43) as $p_0 - \bar{E}(1 + 2\gamma \cos(2kx + \theta) + \gamma^2)$.

It varies sinusoidally along the x-axis. If the beam were infinite, no radiation pressure would result on an infinite surface, as has already been pointed out. The change in the mean pressure is exactly compensated at any x by the flux of momentum given by Eq. (2.44). The resultant stress \bar{T}_{xx} equals p_0 as in the case where $\gamma = 0$.

If the beam is of finite section traversing a liquid with hydrostatic pressure p_0 the surrounding liquid still has the tendency to raise the pressure inside the beam to p_0 . But a uniform pressure $\bar{p}_t = p_0$ along the beam is not compatible with the differential equation (2.14) of our problem, except for the special case in which $\gamma = 0$ (Sec. 7a). This can also be seen from the solution for \bar{p} in Eq. (2.43). When a reflected wave exists, \bar{p}_t necessarily varies periodically in space along the x-axis. It is reasonable to conclude therefore that by action of the surrounding medium the average value in space of the total pressure $\bar{p}_t = \bar{p} + P_0$ is brought to p_0 ; inside the beam \bar{p}_t varies periodically along the x-axis around the new value p_0 .

At points remote from the beam the pressure in the undisturbed medium is assumed to be homogeneous and equal to p_0 . Within an "edge-region" of the beam a transition takes place from the periodically varying pressure \bar{p}_t , which is postulated to exist in the beam, to the homogeneous static pressure p_0 in the surrounding medium. In this transient zone a more complicated (rotational) motion of particles will occur. A closer theoretical investigation of this effect is beyond our scope; but we may reason that the actual transient zone will be small if the wavelength inside the beam is a small fraction of the width and length of the beam.

Our conclusion that the space-average in \bar{p}_t inside the beam will become identical with the static pressure p_0 outside the beam (strictly at $y = z = \infty$), can be based on the following theoretical consideration: Taking the time average of the differential equation (2.14), and assuming a periodic solution in time, we get

$$\frac{\partial}{\partial x} (\bar{p} + \overline{\rho u^2}) = \frac{\partial}{\partial x} (\bar{p} + \overline{\rho u^2}) = 0$$

or

$$\bar{p} + \overline{\rho u^2} = \text{Constant} \quad (2.51)$$

This constant is obviously independent of t and x . But at small amplitudes it is also independent of the amplitude ξ_0 of the wave motion, or of the quantity $k\xi_0$, since the sum $(\bar{p} + \overline{\rho u^2})$ is independent of $k\xi_0$ as seen by adding Equations (2.43) and (2.44)*. In the case of a plane wave of infinite width we find the constant to be zero. For a finite beam within an undisturbed medium, however, the constant becomes $\bar{E} (1 + \gamma^2)$. Indeed, if we assume that the amplitude is gradually decreased, that is $k\xi_0 \rightarrow 0$ or $u \rightarrow 0$, the total pressure \bar{p}_t averaged in time and space along the x -axis must tend towards p_0 , the static pressure in the surrounding medium. Therefore for $u \rightarrow 0$

$$\bar{p}_t = \frac{1}{\lambda} \int_a^{a+\lambda} \bar{p}_t dx = p_0 + \frac{1}{\lambda} \int_a^{a+\lambda} \bar{p} dx \rightarrow p_0$$

or

$$\left[\bar{p} \right]_{u \rightarrow 0} = \left[\frac{1}{\lambda} \int_a^{a+\lambda} \bar{p} dx \right]_{u \rightarrow 0} = 0$$

* See also Part II, Sec. 9.

From Eq. (2.43) we find that we have to add the constant $\bar{E} (1 + \gamma^2)$ to \bar{p} in order to make its mean value in space vanish. (Any additive constant in \bar{p} is compatible with the solution of Eq. (2.14)). Thus under the action of the surrounding medium the pressure \bar{p} becomes

$$\bar{p} = -\bar{E} (1 + 2\gamma \cos(2kx + \theta) + \gamma^2) + \bar{E} (1 + \gamma^2)$$

or
$$\left[\bar{p} = -2\bar{E} \gamma \cos(2kx + \theta) \right]_{u \rightarrow 0}$$

The Constant $= \bar{p} + \overline{\rho u^2}$ now reaches the value $\bar{E} (1 + \gamma^2)$, as seen from Eq. (2.44), in agreement with our statement above.

Since the constant is independent of the amplitude ξ_0 , the foregoing conclusion holds not only for $u \rightarrow 0$, but for any (small) value of u . The mean total pressure $\frac{1}{\lambda} \int_a^{a+\lambda} \bar{p}_t dx$ is therefore equal to p_0 ; that is the mean value of \bar{p} in space is zero, in the case under consideration.

The resultant stress-tensor is obtained therefore from Eq. (2.45) by increasing the mean pressure in space by the amount $\bar{E} (1 + \gamma^2)$:

$$\bar{T} = \begin{vmatrix} -(p_0 + \bar{E}(1 + \gamma^2)) & 0 & 0 \\ 0 & -(p_0 - 2\gamma \bar{E} \cos(2kx + \theta)) & 0 \\ 0 & 0 & -(p_0 - 2\gamma \bar{E} \cos(2kx + \theta)) \end{vmatrix} \quad (2.52)$$

The stress tensor, averaged in time and space, becomes

$$\bar{T} = \begin{vmatrix} -(p_0 + \bar{E}(1 + \gamma^2)) & 0 & 0 \\ 0 & -p_0 & 0 \\ 0 & 0 & -p_0 \end{vmatrix} \quad (2.53)$$

This leads, using the same considerations as above, to a radiation pressure upon a reflecting surface, whose reflection coefficient is γ , of the amount

$$\bar{P}_{\text{rad}} = \bar{E} (1 + \gamma^2) \quad (2.54)$$

For a perfectly reflecting surface ($\gamma = 1$) we have the well known result

$$\left(\bar{P}_{\text{rad}} \right)_{\gamma=1} = 2 \bar{E} \quad (2.55)$$

In the general case of a reflecting surface characterized by the amplitude reflection coefficient γ and the phase angle θ of the reflected wave, the radiation pressure is independent of this angle θ . Here too only energy densities are involved, as $\gamma^2 \bar{E}$ represents the mean energy density of the reflected wave component. The resulting equation (2.54) can therefore be described by saying that the radiation pressure is composed of two parts: One part is due to the incident wave with the energy density \bar{E} , which we may assume to be perfectly absorbed by the surface. This leads to a term $\bar{P}_{\text{rad } 1} = \bar{E}$. The other part is due to a reflected wave with the energy density $\gamma^2 \bar{E}$, which we may imagine as re-emitted by the surface; this causes a pressure $\bar{P}_{\text{rad } 2} = \gamma^2 \bar{E}$. The whole pressure $\bar{P}_{\text{rad}} = (1 + \gamma^2) \bar{E}$ may therefore be regarded as if the surface struck by the plane wave absorbed its energy perfectly and reemitted the amount $\gamma^2 \bar{E}$ (see ref. 5, p. 294). This statement gives of course not a description of what actually happens. A physical explanation of the actual process at the

reflector is offered in Sec. 12 below.

c. Radiation pressure at oblique incidence

If the plane wave strikes a wall inclined at an angle α with respect to the direction x of the wave propagation (Fig. 4, Page 7) the radiation pressure upon this wall is found by transforming the fixed coordinate system $x y z$ into the system $x' y' z$ (Fig. 5, Page 8) whose $x'y'$ axes are rotated through the angle α . Applying the transformation formula (1.13) for the tensor T , we get the new tensor components in the $x' y' z$ system. From (1.13) and (2.49) we find for a finite beam traversing a liquid with static pressure p_0 and striking a perfectly absorbing surface:

$$\bar{T}' = \begin{vmatrix} -(p_0 + \bar{E} \cos^2 \alpha) & \frac{\bar{E}}{2} \sin 2\alpha & 0 \\ \frac{\bar{E}}{2} \sin 2\alpha & -(p_0 + \bar{E} \sin^2 \alpha) & 0 \\ 0 & 0 & p_0 \end{vmatrix} \quad (2.56)$$

Fig. 10 shows the stresses exerted upon a volume element oriented according to the new axes $x' y' z$.

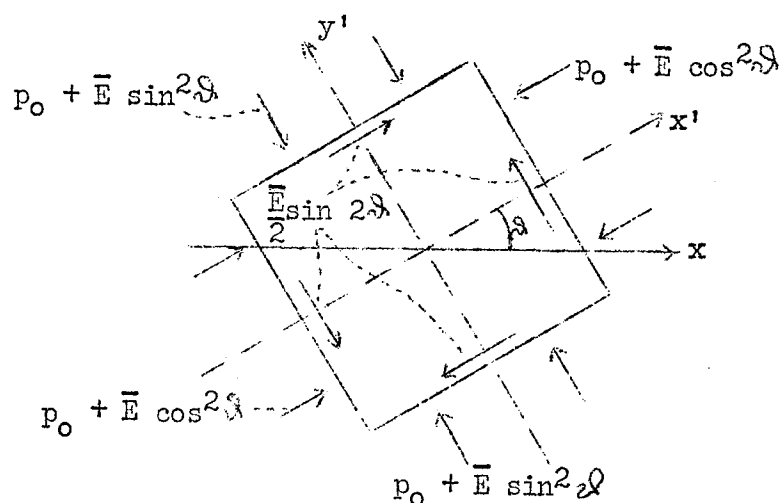


Fig. 10

Components of stress due to radiation at a volume element inclined at an angle α

If we imagine a material surface inserted in the way of the beam at an angle α to be adjacent to the volume element in Fig. 10 and parallel to the y' -axis, it undergoes a pressure $p_0 + \bar{E} \cos^2 \alpha$ in the direction of its normal x' , since it has to exert the same pressure upon the $y'z$ surface of the volume element. The radiation pressure normal to the perfectly absorbing surface is therefore given by $\bar{E} \cos^2 \alpha$. If we assume a reflecting surface, we have to add the radiation pressure exerted by the reflected wave regarded as emitted in the direction shown in Fig. 11.

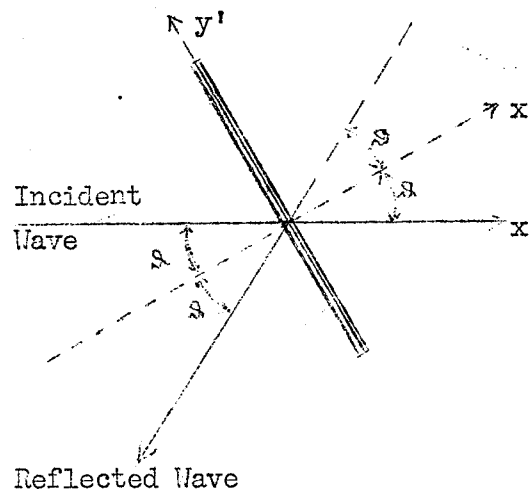


Fig. 11

Direction of propagation of the reflected wave
for oblique incidence

As can easily be seen, we have only to change α in Eq. (2.56) into $-\alpha$ to get the stress components related to the reflected wave.

The pressure exerted by the reflected wave normal to the reflecting surface is therefore $\gamma^2 \bar{E} \cos^2 \alpha$. The total radiation pressure normal to the reflecting surface amounts to

$$\left(\bar{P}_{\text{rad}} \right)_{x'} = (1 + \gamma^2) \bar{E} \cos^2 \alpha \quad (2.57)$$

As Eq. (2.56) shows, the wave exerts also a shearing force upon the reflector, if the latter is inclined at an angle α ; it amounts to $\frac{\bar{E}}{2} \sin 2\alpha$ and therefore has its maximum at $\alpha = 45^\circ$.

Owing to the reversal in the sign of α for the reflected wave, the shearing force due to this wave is $-\gamma^2 \frac{\bar{E}}{2} \sin 2\alpha$. The resultant shearing force per square centimeter connected with the normal radiation pressure in (2.57) is therefore

$$\left(\bar{S}_{\text{rad}}\right)_{y'} = (1 - \gamma^2) \frac{\bar{E}}{2} \sin 2\alpha \quad (2.57a)$$

It vanishes for a perfect reflector ($\gamma = 1$) or for $\alpha = 0$.

It may be noted that the reflection coefficient γ is a complicated function of the angle of incidence α , because a compressional wave striking the wall at the angle α induces in general compressional and rotational waves inside the reflector.¹⁰ Our object here is limited to the knowledge of the average value in time of the resultant second order forces acting upon the reflector and due to radiation pressure; the reflector is assumed to be characterized by the amplitude reflection coefficient $\gamma(\alpha)$, and to absorb all the energy not reflected, so that no radiation leaves the rear of the reflector. In the special case of a perfect reflector $\gamma = 1$ and (by definition) is independent of α .

In the case where some of the energy is transmitted through the reflector, we can assume that the reflector radiates this part from its rear face. The reactional force is proportional to the energy passing through the reflector, and of opposite sign to the forces of both the incident and the reflected wave at the front side. If we call σ the amplitude transmission-coefficient

of the wave leaving the rear of the surface, the resulting radiation pressure is obviously given by

$$\bar{P}_{\text{rad}} = (1 + \gamma^2 - \delta^2) \bar{E} \quad (2.58)$$

for a wave at normal incidence. For oblique obstacles the corresponding forces can be found in the same way.

As an example of the action of the radiation pressure upon oblique surfaces, we compute the force exerted upon the device shown in Fig. 12, which is used frequently for measuring radiation pressure.

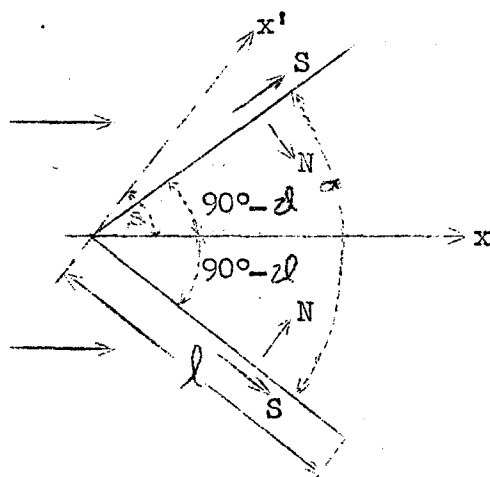


Fig. 12

Normal (N) and shearing (S) forces at a wedge undergoing radiation pressure

The width of the vane in Fig. 12 is l , the dimension perpendicular to the plane of the paper is h , the reflection coefficient γ .

The whole force normal to one side of the vane is $N = lh(1 + \gamma^2) \bar{E} \cos^2 \alpha$, the shearing force is $S = lh(1 - \gamma^2) \frac{\bar{E}}{2} \sin 2\alpha$. (We

neglect the diffraction effect induced by the edges of the vane and effects due to the hydrodynamic flow). As we are interested in

the resultant force in the direction x of wave propagation, we find (Fig. 12):

$$\begin{aligned}\bar{P}_x &= 2 S \cos(90^\circ - 2\ell) + 2 N \cos 2\ell = 2(S \sin 2\ell + N \cos 2\ell) \\ &= 2 \ell h \left\{ (1 - \gamma^2) \frac{\bar{E}}{2} \sin 2\ell \sin 2\ell + (1 + \gamma^2) \bar{E} \cos^2 2\ell \cos 2\ell \right\} \\ &= 2 \ell h \bar{E} \cos 2\ell (1 + \gamma^2 \cos 2\ell)\end{aligned}$$

Introducing the angle $\alpha = 2(90^\circ - 2\ell) = \pi - 2\ell$ included between the two sides of the vane, and calling attention to the fact that γ is a function of α by writing $\gamma(\alpha)$ instead of γ , we have:

$$\bar{P}_x = 2 \ell h \bar{E} \cdot \sin \frac{\alpha}{2} (1 - \gamma^2(\alpha)) \cdot \cos \alpha \quad (2.59)$$

$\alpha = 180^\circ$ corresponds to a plane surface normal to the x-axis and from (2.58) we find as expected, $\bar{P}_x = 2 \ell h \bar{E} (1 + \gamma^2)$. The force at the vane tends toward zero with decreasing α .

It might be interesting to note that for $\alpha = 90^\circ$ \bar{P}_x becomes independent of $\gamma(\pi/2)$ and equals $(\bar{P}_x)_{90^\circ} = \sqrt{2} \ell h \bar{E}$. For every value of γ , a 90° -vane undergoes the same force (\bar{P}_x) as a perfect absorber presenting an area $\sqrt{2} \ell h$ normal to the incident wave. Since the reflected wave propagates in this case in the direction normal to the x-axis, it carries no mean flux of momentum along the x-direction. This explains the independence of $(\bar{P}_x)_{90^\circ}$ from the reflection coefficient $\gamma(\pi/2)$.

8. Radiation Pressure on a Perfect Absorber at Finite Amplitudes

All previous considerations are valid only for small amplitudes, where $k\xi_0 \ll 1$, because we used a development in series of

the exact solution for u and p , retaining terms up to the second order only. But since we have the exact solutions for liquids in Eqs. (2.30) and (2.31), we are able to deal also with finite amplitudes. We limit our consideration of finite amplitudes here to the case of a wave traveling in one direction only, terminating at a perfect absorber. This example will show clearly enough the difference between the physical characteristics of small and finite amplitudes.

For computing the tensor components in Eq. (2.18) we need the mean pressure \bar{p} and the mean flux of momentum $\overline{\rho u^2}$. Using the solutions (2.30) and (2.31) for u and p with $\gamma = 0$, and the density ρ from (2.28), we find

$$\bar{p} = \frac{1}{T_p} \int_0^{T_p} \frac{k\xi_0}{\beta} \cos(\omega t - kx + k\xi_0 \sin(\omega t - kx)) dt$$

$$\overline{\rho u^2} = \frac{1}{T_p} \int_0^{T_p} \frac{\omega^2 \rho_0 \xi_0^2 \cos^2(\omega t - kx + k\xi_0 \sin(\omega t - kx))}{1 - k\xi_0 \cos(\omega t - kx + k\xi_0 \sin(\omega t - kx))} dt$$

Introducing $\omega^2 \rho_0 = k^2/\beta$ and the new variable $\omega t - kx = \tau$, we have

$$\bar{p} = \frac{k\xi_0}{\beta} \cdot \frac{1}{2\pi} \int_0^{2\pi} \cos(\tau + k\xi_0 \sin \tau) d\tau \quad (2.60)$$

$$\overline{\rho u^2} = \frac{k^2 \xi_0^2}{\beta} \cdot \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos^2(\tau + k\xi_0 \sin \tau)}{1 - k\xi_0 \cos(\tau + k\xi_0 \sin \tau)} d\tau \quad (2.61)$$

The integral in (2.60) is found to be represented by the Bessel-function of index 1, $-J_1(k\xi_0)$. So we have

$$\bar{p} = -\frac{k\xi_0}{\beta} J_1(k\xi_0) \quad (2.62)$$

For small values of $k\xi_0$ ($k\xi_0 \ll 1$) the function J_1 is given by

$$J_1(k\xi_0) = \frac{k\xi_0}{2} - \frac{(k\xi_0)^3}{16} + \dots$$

Taking the first term, we get for small amplitudes from (2.62)

$$-\bar{\beta p} = k\xi_0 \cdot \frac{k\xi_0}{2} = \frac{k^2 \xi_0^2}{2} = \frac{\beta \omega^2 \rho_0 \xi_0^2}{2} = \beta \bar{E}$$

according to Eq. (2.42). For larger amplitudes the mean excess pressure \bar{p} deviates from $-\bar{E}$, since p is represented by (2.62).

We find:

$$\begin{array}{cccccccc} k\xi_0 & = & 0.05 & 0.1 & 0.3 & 0.5 & 0.7 & 0.8 & 0.9 & (1.0) \\ -\beta p & = & 0.0011 & 0.005 & 0.044 & 0.121 & 0.230 & 0.295 & 0.365 & (0.440) \end{array} \quad (A)$$

As $k\xi_0$ approaches 1, the values given here for $-\beta p_0$ are only of theoretical interest since the concept of constant compressibility loses its meaning at such high compressions as are encountered at larger values of $k\xi_0$.

The expression (2.61) for $\overline{\rho u^2} = 2 \bar{E}_{\text{kin}}$ cannot be represented by known functions in a closed simple form, as is the case with \bar{p} .

The integral

$$F(k\xi_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\cos^2(\tau + k\xi_0 \sin \tau)}{1 - k\xi_0 \cos(\tau + k\xi_0 \sin \tau)} d\tau \quad (2.63)$$

has been evaluated numerically for different values of the parameter $k\xi_0$.* The result is as follows:

* The writer is grateful to Dr. S. P. Frankel of this institution for suggesting a quick and simple method for evaluating the integral in Eq. (2.63).

$$\begin{array}{l}
 k\xi_0 = 0.05 \quad 0.1 \quad 0.3 \quad 0.5 \quad 0.7 \quad 0.8 \quad 0.9 \quad 1.0 \\
 F(k\xi_0) = 0.500 \quad 0.5025 \quad 0.524 \quad 0.571 \quad 0.685 \quad 0.786 \quad 1.057 \quad \infty
 \end{array} \quad (B)$$

For small values of $k\xi_0$ the integral $F(k\xi_0)$ equals $1/2$ and

$\overline{\rho u^2} = k^2 \xi_0^2 / 2\beta = \bar{E}$ according to Eq. (2.61). For increasing values of $k\xi_0$ the value of $\overline{\rho u^2}$ undergoes a deviation from the energy density \bar{E} . We obtain for $\beta \cdot \overline{\rho u^2} = (k\xi_0)^2 \cdot F(k\xi_0)$:

$$\begin{array}{l}
 k\xi_0 = 0.05 \quad 0.1 \quad 0.3 \quad 0.5 \quad 0.7 \quad 0.8 \quad 0.9 \quad (1.0) \\
 \beta \cdot \overline{\rho u^2} = 0.001 \quad 0.005 \quad 0.047 \quad 0.143 \quad 0.336 \quad 0.503 \quad 0.856 \quad (\infty)
 \end{array} \quad (C)$$

The quantity $\bar{p} + \overline{\rho u^2}$ changes thus with increasing amplitudes:

$$\begin{array}{l}
 k\xi_0 = 0.05 \quad 0.1 \quad 0.3 \quad 0.5 \quad 0.7 \quad 0.8 \quad 0.9 \quad (1.0) \\
 \beta \cdot (\bar{p} + \overline{\rho u^2}) = 0.000 \quad 0.000 \quad 0.003 \quad 0.021 \quad 0.106 \quad 0.208 \quad 0.491 \quad (\infty)
 \end{array} \quad (D)$$

For smaller values of $k\xi_0$ the following series developments may be used:

$$\beta \bar{p} = -\frac{1}{2}(k\xi_0)^2 + \frac{1}{16}(k\xi_0)^4 - \frac{1}{384}(k\xi_0)^6 + \frac{1}{18432}(k\xi_0)^8 - \dots \quad (2.62)$$

$$\beta \overline{\rho u^2} = +\frac{1}{2}(k\xi_0)^2 + \frac{1}{4}(k\xi_0)^4 + \frac{13}{96}(k\xi_0)^6 + \frac{761}{3072}(k\xi_0)^8 + \dots \quad (2.61a)$$

The development of $\beta \bar{p}$ follows immediately from (2.62) and the well-known series development of the Bessel-function $J_1(x)$. The development of $\beta \overline{\rho u^2}$ was obtained by developing the integrant in (2.61) in powers of $k\xi_0 \cos(\tau + k\xi_0 \sin \tau)$, converting the powers of the cosines into cosines of multiples of the argument, then integrating term by term by using the relation

$$(1/2\pi) \int_0^{2\pi} \cos n(\tau + k\xi_0 \sin \tau) d\tau = (-1)^n J_n(nk\xi_0),$$

and finally developing the Bessel-functions $J_n(nk\xi_0)$ in powers of $nk\xi_0$. The physical conclusions to be drawn from these results are as follows:

a) Infinitely extended plane compressional waves,
not communicating with undisturbed regions

For small amplitudes, as table (D) shows us again, the two terms \bar{p} and $\overline{\rho u^2}$ cancel each other, because both quantities equal the mean energy-density $-\bar{E}$ or $+\bar{E}$. At increasing amplitudes the sum $(\bar{p} + \overline{\rho u^2})$, which according to (2.18) is responsible for the radiation pressure in this case, is positive. At higher amplitudes, therefore, even in the case of an infinitely extended plane wave in a non-absorbing medium, a radiation pressure exists, though its value is negligible if the amplitudes are not extremely high. In the limit, at which $k\xi_0 \rightarrow 1$, this radiation pressure would theoretically tend towards infinity. This can be understood physically by the fact that for $k\xi_0 = 1$ the density becomes infinite once in each cycle, leading to infinite values of the double mean kinetic energy-density $\overline{\rho u^2}$. But at such great amplitudes the concept of a constant compressibility will surely not hold. Therefore these statements concerning extremely large amplitudes ($k\xi_0 \rightarrow 1$) are only of theoretical interest.

b) Finite plane beam surrounded by or communicating
with undisturbed regions and incident on a perfect
absorber

In this case, as stated above, only the term $\overline{\rho u^2}$ is responsible for the radiation pressure, as the change in pressure \bar{p} will be equalized by the hydrostatic pressure p_0 outside the finite beam. For small amplitudes, as may be seen from Eq. (2.61) and the foregoing table (C), and as is known from previous considerations, $\overline{\rho u^2} = \bar{E}$. That is, as a first approximation, the radiation pressure equals the mean total energy density \bar{E} . At higher ampli-

tudes this is no longer true.

The general statement, correct for any amplitude, in this case is as follows:

The radiation pressure exerted by a compressional plane wave which is surrounded by or in communication with* a medium not affected by the wave motion and in contact with the rear of a totally absorbing surface, equals $2 \bar{E}_{kin}$ when \bar{E}_{kin} is the mean kinetic energy-density of the plane wave in Eulerian coordinates. This is correct for any amplitude, since $\overline{\rho u^2} = 2 \bar{E}_{kin}$ by definition. With this in mind, the formula already given in Eq. (1.17) can be regarded as generally correct.

As to mean energy densities at finite amplitudes, one must bear in mind that \bar{E}_{pot} and \bar{E}_{kin} and also \bar{E}_{pot}^* and \bar{E}_{kin}^* , are not the same in Eulerian as in Lagrangian coordinates. This disparity disappears at small amplitudes, that is, in the second order of approximation. The acoustic intensity is given by $J = p^*u^*$, using our previous consideration concerning a perfect absorber (page 41). The mean intensity at any amplitude is therefore given by

* Such a communication can be regarded as realized for instance by a small hole in the absorbing surface, on the front of which is the incident beam, while the rear face is in contact with an undisturbed region of static pressure p_0 . As G. Richter¹⁵ has already correctly remarked, the hole should be visualized as opaque to acoustic radiation but as allowing the equalization of average pressure between the two regions, as might be effected by having a small totally absorbing piston free to move in the hole.

$$\bar{J} = \frac{1}{T} \int_0^T p^* u^* dt = \frac{\omega k \xi_0^2}{\beta} \cdot \frac{1}{2\pi} \int_0^{2\pi} \cos^2(\omega t - kx) d\omega t$$

$$\text{or } \bar{J} = c \cdot \frac{k^2 \xi_0^2}{2\beta} = c \overline{u^{*2}} = 2 c \bar{E}_{\text{pot}}^*$$

using Eqs. (2.24), (2.25), (2.40) and $c^2 \beta \rho_0 = 1$. The amplitudes belonging to a mean intensity \bar{J} are therefore always given by

$$k \xi_0 = \sqrt{\frac{2 \beta \bar{J}}{c}}$$

If we assume radiation in water at 20° C, with $k \xi_0 = 1$ (a purely hypothetical value), the intensity \bar{J} is found to be

$$\bar{J} = \frac{c}{2\beta} = \frac{1.49(10^5)}{2 \cdot 46(10^{-6})} \frac{\text{cm kg}}{\text{sec. cm}^2} = 1.62(10^9) \cdot \frac{9.81}{100} \frac{\text{watt}}{\text{cm}^2}$$

$$\text{or } \bar{J} = 1.59(10^8) \frac{\text{watt}}{\text{cm}^2}$$

This expression shows how far we always are experimentally from "finite" amplitudes, because even an intensity of 100 watt cm⁻² is to be regarded as very high in water. The value of $k \xi_0$ for an intensity $\bar{J} = 100$ watt cm⁻² is found to be 7.93(10⁻⁴).

For all practical purposes in liquids the amplitude as well as the quantity $k \xi_0$ can be regarded as small. The radiation pressure is therefore described for all cases which are experimentally feasible by the considerations in Section 7, where we dealt with small amplitudes.

Nevertheless the theoretical results for finite amplitudes are not without interest, since they show that the relation $\bar{P}_{\text{rad}} = \text{mean total energy density } \bar{E}$ cannot be regarded as a "basic" physical law as it is considered to be in electrodynamics. For

mechanical wave motions a law of this kind holds only at small amplitudes, whereas in general at all amplitudes, the radiation pressure is linked with the expression for the mean density of kinetic energy in Eulerian coordinates.

As $\bar{E} = \bar{E}_{\text{kin}} + \bar{E}_{\text{pot}}$, the difference $\bar{E} - 2\bar{E}_{\text{kin}} = \bar{E}_{\text{pot}} - \bar{E}_{\text{kin}} = 1/2 (\bar{\rho}_0 u^2 - \bar{\rho} u^2)$ in agreement with Eq. (2.40). In liquids this difference turns out to be small of the sixth order in $(k\xi_0)$. In order to show this, one uses the series development of $\bar{\rho} u^2$ as given by Eq. (2.62); $\bar{\rho}_0 u^2$ is found with the aid of (2.30) to be represented by $1/2 (k\xi_0)^2 [1 + J_2(2k\xi_0)]$. Making use of the known series development of $J_2(x) = x^2/2 - x^4/24 + \dots$, we obtain indeed $\bar{\rho} (\bar{E} - 2\bar{E}_{\text{kin}}) = - (k\xi_0)^6/64 + \dots$, that is, only terms of the sixth order and higher are involved.

9. Note on an Integral of the Eulerian Equation of Motion and the Flow of Mass for Infinitely Extended Plane Waves in Liquids

For liquids with constant compressibility we found that the solutions of the Lagrangian equation (2.22) and consequently those of the Eulerian equation (2.14) were functions of the argument $(\omega t \mp kx)$ only. Using the fact that therefore $\partial/\partial t = \mp c \cdot \partial/\partial x$, Eq. (2.14) can be integrated with respect to x or t and we obtain

$$\mp c \cdot \rho u + \rho u^2 + p = C \quad (2.64)$$

C is independent of x and t , but not necessarily of the amplitude of the wave motion. We can prove, however, that C has the value zero. This proof, which we will now give, verifies at the same time the fact, already used in Part II, Sec. 5, that any function

$f(\omega t \mp kx)$, which was found to satisfy the Lagrangian equation (2.22), leads to an exact solution of the corresponding Eulerian equation (2.14), when the transformations of the quantities u^* , ρ^* , p^* into the corresponding Eulerian quantities u , ρ , p are properly made.

Let $\partial \xi / \partial t = u^* = u^*(\omega t \mp kx)$ be the general solution in the Lagrangian system.* From Eqs. (2.10) and (2.28) we obtain

$$p^* = -\frac{1}{\beta} \frac{\partial \xi}{\partial x} = \pm \frac{1}{\beta c} \frac{\partial \xi}{\partial t} = \pm \rho_0 c \cdot u^* \quad (2.65a)$$

$$\rho^* = \frac{\rho_0}{1 + \frac{\partial \xi}{\partial x}} = \frac{\rho_0}{1 \mp \frac{u^*}{c}} \quad (2.65b)$$

Transforming p^* , ρ^* and u^* into p , ρ and u by replacing the variable x in u^* by $(x - \xi)$ and inserting p and ρ , as found in Eqs. (2.65a, b), into Eq. (2.64), we have

$$\mp c \cdot \frac{\rho_0 u}{1 \mp \frac{u}{c}} + \frac{\rho_0 u^2}{1 \mp \frac{u}{c}} \pm \rho_0 c u = C$$

or $\underline{C = 0}$

Thus, Eq. (2.64) provides us with the general relation, valid in liquids at any time t and at any x :

$$p + \rho u^2 = \pm c \cdot \rho u \quad (2.66)$$

Averaging (2.66) in time, the mean mass flow carried by the wave

* $u^*(\omega t \mp kx)$ stands for $u_1^*(\omega t - kx) + u_2^*(\omega t + kx)$, where $u_{1,2}^*$ are arbitrary functions of the arguments $(\omega t \mp kx)$.

motion through a fixed unit cross-section turns out to be

$$\overline{\rho u} = \pm \frac{1}{c} (\bar{p} + \overline{\rho u^2}) \quad (2.66a)$$

The plus or minus sign corresponds to waves traveling to the right or left. The value of $(\bar{p} + \overline{\rho u^2})$ at finite amplitudes has been given in Table (D), p. 57. From Eqs. (2.60) and (2.61) we obtain in the case of sinusoidal wave motion:

$$\frac{1}{c} (\bar{p} + \overline{\rho u^2}) = \frac{k\xi_0}{2\pi\beta c} \int_0^{2\pi} \frac{\cos(\tau_{\pm} \pm k\xi_0 \sin \tau_{\pm})}{1 \mp k\xi_0 \cos(\tau_{\pm} \pm k\xi_0 \sin \tau_{\pm})} d\tau \quad (2.67)$$

The series development of the integral in powers of $(k\xi_0)$, is found by adding Eqs. (2.62a) and (2.61a) on page 57:

$$\overline{\rho u} = \pm \frac{1}{c} (\bar{p} + \overline{\rho u^2}) = \pm \rho_0 c \left(\frac{5}{16} (k\xi_0)^4 + \frac{17}{128} (k\xi_0)^6 + \dots \right) \quad (2.68)$$

Eq. (2.68) brings once more to evidence the result formerly obtained, that for small amplitudes, that is disregarding terms of higher than second power, the sum $\bar{p} + \overline{\rho u^2}$ cancels out. It confirms also the conclusion made concerning the "Constant" in Section 7b, namely that this constant is independent of $(k\xi_0)$ to the second order of approximation.

The series development of the change in average density due to the wave motion $\overline{\Delta \rho} = \bar{\rho} - \rho_0$ follows likewise immediately from (2.68). Eq. (2.65b) shows that $\rho^* - \rho_0 = \pm \rho^* u^*/c$. Transforming this into Eulerian coordinates and taking the time average we obtain

$$\overline{\Delta \rho} = \bar{\rho} - \rho_0 = \pm \overline{\rho u}/c = \rho_0 \left(\frac{5}{16} (k\xi_0)^4 + \frac{17}{128} (k\xi_0)^6 + \dots \right) \quad (2.68a)$$

As stated on page 44 the change in mean density $\overline{\Delta \rho}$ vanishes to the second-order approximation; it is of the fourth order in $(k\xi_0)$ and positive, whereas the corresponding change in pressure $\bar{p} = -\bar{E}$ is of the second order and negative.

The average flow of mass $\overline{\rho u}$ vanishes to the second order of approximation, as stated in various papers (see for instance reference 16). This does not mean, however, that it is strictly zero as is seen from Eq. (2.68)*. In liquids, the values of $k\xi_0$ that can be attained experimentally are very small, and the quantity $\overline{\rho u}$ amounts numerically to an extremely small value. It may be noted that the existence of a "net flow of mass" does not necessarily imply an actual transport of matter, that is, a mean particle displacement. Indeed, in a pure sinusoidal wave-motion every particle is found at exactly the same place as before after a full period. The average values in time of ξ or u^* are zero in this case. Nevertheless, owing to the fact that ρ is not a constant, but $\rho = \rho_0 + \rho_1 + \rho_2 + \dots$, ($\rho_1, \rho_2 \dots$ denoting higher-order terms in ρ , varying sinusoidally in time) the time-average of the "flux of mass-density", given by $\overline{(\rho \cdot u)}$, may have a value different from zero.

The quantity $\overline{\rho u}$ is independent of x . To prove this, we take the time-average of the Eulerian equation (2.14) and find for a wave motion periodic in time,

$$\overline{\frac{\partial}{\partial x} (p + \rho u^2)} = \frac{\partial}{\partial x} (\bar{p} + \overline{\rho u^2}) = 0$$

* The only exception, where $\overline{\rho u}$ equals zero strictly, is the case of a standing wave.

or, according to Eq. (2.66a), $\partial \overline{\rho u} / \partial x = 0$. Consequently $\overline{\rho u}$ is independent of x and a function of the wave amplitude $(k\xi_0)$ only.

Thus we have in liquids at any amplitude for an infinitely extended plane wave

$$\bar{p} + \overline{\rho u^2} = \pm c \cdot \overline{\rho u} = \rho_0 c^2 \cdot f(k\xi_0) \quad (2.69)$$

with

$$f(k\xi_0) = \frac{5}{16} (k\xi_0)^4 + \frac{17}{128} (k\xi_0)^6 + \frac{4567}{18432} (k\xi_0)^8 + \dots$$

10. Radiation Pressure and Viscous Absorption

The Eulerian differential equation (2.14) includes the case of plane compressional waves in viscous fluids. But in the case of viscous absorption an additional term, associated with the coefficient of viscosity η , enters the differential equation both in the Eulerian form Eq. (2.14) and the Lagrangian form Eq. (2.22). D'Alembert's general solution $f(\omega t \mp kx)$ is then no longer a solution of Eq. (2.22), as is well known, and we cannot make use of the relation $\partial / \partial t = \mp c \cdot \partial / \partial x$. The conclusions based on this relation in the foregoing section 9 therefore do not hold for viscous liquids. Still, taking the average value in time of Eq. (2.14), as we did at the end of section 9, we find also in the case of viscous liquids (or gases) that $\partial (\bar{p} + \overline{\rho u^2}) / \partial x = 0$ for a wave motion periodic in time and therefore $\bar{p} + \overline{\rho u^2} = \text{constant}$. As the amplitude of a unidirectional plane wave decreases steadily along its direction of propagation through a viscous medium, the values of \bar{u} and \bar{p} steadily approach zero. Thus the value of the constant is necessarily zero and \bar{p} and $\overline{\rho u^2}$ cancel each other in the same way as they do in absence of viscosity. The radiation

pressure $\bar{p} + \overline{\rho u^2}$ due to a beam of infinite width (and of any amplitude) is therefore zero at a perfect absorber in the case of a viscous liquid (or fluid). The same conclusion holds for a reflected plane wave leaving an absorber.

For a beam of finite width the problem becomes more involved, owing to the interaction of the beam with undisturbed regions. For non-viscous liquids, the time-average pressure \bar{p} is constant along the beam, or, even when periodic in space, still its average value along the beam is constant. This fact has enabled us to deal with this interaction in a relatively simple manner.

On the other hand in viscous liquids, the value of \bar{p} in the beam changes exponentially along the beam, and the interaction with the undisturbed region outside the beam involves rotational motion of the fluid. The problem of radiation pressure becomes closely linked with that of the hydrodynamic flow. A very complete treatment of the forces due to second order effects in viscous media is given in the fundamental paper by Eckart.⁶

11. Rayleigh Pressure and Langevin Pressure in Liquids

In papers dealing with acoustic radiation pressure special expressions have recently been introduced in the literature to denote the special circumstances under which radiation pressure may be observed. It is our purpose to clarify the physical meaning of these usages. According to our previous considerations we can distinguish four different cases of pressures; they are connected on the one hand with the coordinate

system used (Eulerian or Lagrangian), and on the other hand with the kind of interaction of the acoustic beam with an undisturbed medium, that is, whether the beam is regarded as of infinite width and not communicating with an undisturbed part of the medium, or of finite cross section and surrounded by the undisturbed medium. We may clarify these cases as follows:

1. (L. - i.): Lagrangian coordinates and infinite plane wave.
2. (L. - c.): Lagrangian coordinates and wave region communicating with undisturbed medium.
3. (E. - i.): Eulerian coordinates and infinite plane wave.
4. (E. - c.): Eulerian coordinates and wave region communicating with undisturbed medium.

We will deal with these cases successively (always assuming normal incidence):

1. (L. - i.) At any point on a surface following the motion of a particle in a liquid, the pressure p^* varies purely sinusoidally (Eq. (2.25)). Its time average value is therefore zero: $\overline{p^*} = 0$. An observer moving together with a particle would register a mean pressure equal to zero. The mean pressure $\overline{p_t}$ in the medium for a stationary observer would be found to be lowered to $p_0 - \overline{E} (1 + \gamma^2)$ at small amplitudes in an infinitely extended beam, where p_0 is the pressure in absence of wave motion. For larger amplitudes the decrease in mean pressure follows from Eq. (2.62).

2. (L. - c.) The mean pressure $\overline{p^*}$ for a moving observer is zero, just as in 1. (L. - i.). For a stationary observer the mean pressure $\overline{p_t} = p_0$, since the lowering of p_0 by the wave motion is now counter-balanced by the action of the medium of static

pressure p_0 which surrounds the beam region.

3. (E. - i.) The mean pressure \bar{p}_t at fixed coordinates is lowered, as already mentioned under 1. (L. - i.), by the amount $-\bar{E} (1 + \gamma^2)$ at small amplitudes. If its value was p_0 in absence of wave motion, then for an infinitely extended plane wave, not communicating with any undisturbed medium, the value is $p_0 - \bar{E} (1 + \gamma^2)$. The radiation pressure \bar{P}_{rad} , which we have been led to identify with $(\bar{p} + \overline{\rho u^2})$ in Eulerian coordinates, becomes zero for small amplitudes. For larger amplitudes numerical values of the radiation pressure can be found from Table (D) on page 57 or from Eq. (2.69).

4. (E. - c.) If the acoustic beam communicates with a medium not affected by the wave motion, the mean pressure \bar{p}_t becomes equalized to the value of the static pressure p_0 of the undisturbed medium. This raises the mean density $\bar{\rho}$ in the beam by $\overline{\Delta \rho} = \bar{E}/c^2 (1 + \gamma^2)$ at small amplitudes, as follows from Eq. (2.48b). The radiation pressure $\bar{P}_{\text{rad}} = \bar{E} (1 + \gamma^2)$ for small amplitudes.

The notation "Rayleigh pressure" ($\bar{P}_{\text{Rayleigh}}$), as used in the literature ^{8,11,15}, means the average excess-pressure, due to the wave motion, which would be noticed by an observer moving with a particle. It is therefore identical with the quantity \bar{p}^* in our notation:

$$\bar{P}_{\text{Rayleigh}} = \bar{p}_t^* - p_0 = \bar{p}^* \quad (2.70)$$

For liquids with constant compressibility, \bar{p}^* and consequently the "Rayleigh-pressure" is zero at all amplitudes. For media that have a more complicated relation $p^*(\rho)$ between pressure

and density - as in gases for instance - \bar{p}^* and therefore the Rayleigh-pressure is different from zero. (See Part III, Sec. 3 and 5)

The expression "Langevin pressure" ($\bar{P}_{\text{Langevin}}$) is used, following Hertz and Mende⁸, for the difference between the pressure \bar{p}^* observed at a moving particle or plane and the mean pressure \bar{p} at fixed coordinates:

$$\bar{P}_{\text{Langevin}} = \bar{p}^* - \bar{p} = \bar{P}_{\text{Rayleigh}} - \bar{p} \quad (2.71)$$

For a plane compressional wave traveling in one direction in a liquid, we found at small amplitudes $\bar{p}^* = 0$ and $\bar{p} = -\bar{E}$, therefore

$$\bar{P}_{\text{Langevin}} = \bar{p}^* - \bar{p} = \bar{E} \quad [k\xi_0 \ll 1] \quad (2.72)$$

This result turns out to be independent of the nature of the medium, that is, independent of the special function $p(\rho)$ connecting pressure and density. This fact has already been stated in various papers^{8,1}. An exact and simple proof is as follows:

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = - \frac{\partial p^*}{\partial x}$$

For small amplitudes we have according to Eq. (2.21)

$$p(x) = p^*(x - \xi) = p^*(x) - \xi \frac{\partial p^*}{\partial x} \pm \dots$$

Therefore, at small amplitudes

$$\overline{p^*(x)} - \overline{p(x)} = \overline{\left(\xi \frac{\partial p^*}{\partial x} \right)} = -\rho_0 \overline{\left(\xi \frac{\partial^2 \xi}{\partial t^2} \right)} \quad (2.73)$$

Now we can write

$$\xi \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial}{\partial t} \left(\xi \frac{\partial \xi}{\partial t} \right) - \left(\frac{\partial \xi}{\partial t} \right)^2$$

and taking the average time-value and regarding ξ as a function periodic in time,

$$\overline{\left(\xi \frac{\partial^2 \xi}{\partial t^2}\right)} = - \overline{\left(\frac{\partial \xi^2}{\partial t}\right)}$$

Thus we have from Eq. (2.73)

$$\overline{p^*(x)} - \overline{p(x)} = + \overline{\rho_0 \left(\frac{\partial \xi}{\partial t}\right)^2} = \overline{\rho_0 u^{*2}} = 2\overline{E}_{\text{kin}}(x) \quad (2.74)$$

For a unidirectional wave motion, $\overline{\rho u^{*2}}$ equals \overline{E} and we have $\overline{P}_{\text{Langevin}} = \overline{p^*} - \overline{p} = \overline{E}$. On the other hand, Eq. (2.74) is perfectly general, holding whether or not a reflected wave is present; but in this case $\overline{\rho u^{*2}}$ varies from point to point and we must take the average value in space of Eq. (2.74) if we wish to introduce the mean total energy density \overline{E} . Doing so, we get from (2.74)

$$\overline{P}_{\text{Langevin}} = \overline{\overline{p^*}} - \overline{\overline{p}} = 2\overline{\overline{E}}_{\text{kin}} = \overline{E} (1 + \gamma^2) \quad \left[k\xi_0 \ll 1 \right] \quad (2.75)$$

According to the proof leading to Eq. (2.75), this general result is independent indeed of the special function $p(\rho)$.

In the literature dealing with radiation pressure it is often the Langevin pressure that is identified with the radiation pressure exerted upon a plane obstacle (see references 8 and 1). From the general result Eq. (2.75) it is concluded that the radiation pressure at small amplitudes equals the energy density in any fluid medium.

Hertz and Mende⁸ seem to have been the first to introduce the expression "Langevin-pressure", as defined in (2.75). Nevertheless, Langevin himself does not seem to have had this quantity $\overline{p^*} - \overline{p}$ (or better $\overline{\overline{p^*}} - \overline{\overline{p}}$) in mind in his proof concerning the con-

nection between radiation pressure and energy density as reported in a paper by P. Biquard². According to Langevin's derivation, as given there, one is led to the conclusion that he identifies radiation pressure with $\overline{p_t^*} - \overline{p_o^*}$, that is the difference between the total pressures associated with moving particles, at two adjacent regions of the medium, one affected by the acoustic wave motion and the other unaffected. If in the latter region the fluid is at rest (as it may be, for instance, behind an opaque reflector), $\overline{p_o^*}$ becomes identical with the static pressure p_o (see also the recent note by P. J. Westervelt¹⁷). In our opinion some conclusions in Biquard's report of Langevin's derivation are open to criticism; moreover we cannot accept the quantity $\overline{p_t^*} - p_o$ as representing the true radiation pressure. Whichsoever of these quantities may be called "Langevin-pressure", none of them agrees with the actual radiation pressure given by $(\bar{p} + \overline{\rho u^2})$. To diminish the confusion already widely spread, we recommend that the term "Langevin radiation pressure" be discarded altogether.

Acoustic radiation pressure should properly be identified with the expression $\bar{p} + \overline{\rho u^2}$ in Eulerian coordinates. This quantity is the one actually measured; it is in fact the resultant force per unit area due to the wave motion. The two components \bar{p} and $\overline{\rho u^2}$ act together; the contribution of each depends on the characteristics of the reflector (that is, on γ). The final sum, at small amplitudes, is always $\bar{E} (1 + \gamma^2)$. (See Sec. 12)

It happens that the value of $\bar{P}_{\text{Langevin}}$, as given by Eq. (2.75), which is associated with an infinitely extended plane wave, equals numerically the actual radiation pressure for a beam of finite width; its identification with the radiation pressure

however is evidently misleading as to the physical concept of this quantity.

12. What is Measured as "Radiation Pressure" Experimentally?

The radiation pressure that we measure experimentally is physically not identical with a "pressure" in the hydrodynamic sense. Though of the same dimensions it is a different physical quantity and is expressed by $\overline{p_t} - p_0 + \overline{\rho u^2}$ for a plane compressional wave at normal incidence. The quantity $\overline{\rho u^2}$ can be interpreted as the average flux of mechanical momentum ρu through a unit area fixed in space.

Let us consider a device D for measuring the radiation pressure. It may contain a reflecting plane surface struck by an incident acoustic beam of finite cross-section. This "receiving" plane is to be regarded as moving together with the immediately adjacent particles of the medium. This movement, the amplitude of which depends on the reflector, is purely sinusoidal and the average position of the receiving surface is identical with its position at rest.

As this receiving surface is a part of the whole device D, it is connected mechanically in some way with D. The whole device D may be regarded as having a large mechanical inertia compared with the inertia of the receiving surface; its center of mass can therefore be treated as practically immovable in space, even when the receiving surface is moving periodically. For simplicity we assume that no acoustic energy leaves from its rear side ; all the non-reflected part of the incident energy is assumed to be absorbed within D.

a. We consider first the case of a perfectly absorbing device D. The acoustic beam of finite cross-section is imbedded in the undisturbed surrounding medium which also encloses the device D. Therefore the mean pressure is the same throughout the medium, including the whole surface of D. The same pressure $p_0 = \bar{p}_t$ acts both at the front and rear of D, so that no resulting force due to a hydrostatic pressure is exerted upon D in this case.

Now let us visualize an imaginary surface S, enclosing exactly the device D but stationary in space (Fig. 13).

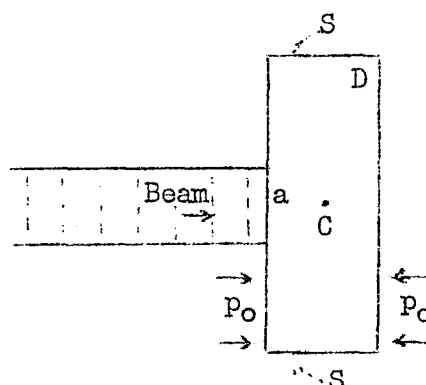


Fig. 13

Schematic diagram of a device D for measuring acoustic radiation pressure; C = center of mass, S = surface fixed in space enclosing the whole device D; a = receiving surface struck by the acoustic beam and mechanically connected with D; p_0 = hydrostatic pressure

All over the receiving part a of D particles are crossing the corresponding part of S according to their periodic movement inside the acoustic beam. The whole force exerted upon the device D, and therefore acting at the center of mass C, equals the gain in momentum of D per second, or the "flux of momentum" through a per second. This latter quantity, as we found, is given by

$\overline{\rho u^2}$ for unit area ($a = 1$). In the small time dt a mass $\rho u \cdot dt$

crosses the unit section of a with the velocity u ; the whole transport of momentum through the section is consequently $\rho u \cdot dt \cdot u = \rho u^2 \cdot dt$. In unit time the flux of momentum amounts, consequently, to $\overline{\rho u^2}$. It may be noticed that both the particles entering D at a , and also the particles leaving D at a somewhat later time contribute to the force exerted upon D in the same sense of direction, the departing particles exerting a reactional force on D . According to our assumption that no radiation leaves the rear of D , no force is exerted at this side upon D . The whole mean force per unit area of D , struck by the compressional wave, therefore equals the time-average value $\overline{\rho u^2}$, calculated for a cross-section fixed in space. This leads us back to the result of our former considerations, that the radiation pressure at a perfect absorber is given by the expression $\overline{\rho u^2}$ in Eulerian coordinates. This double mean kinetic energy density $\overline{\rho u^2}$ equals at small amplitudes the mean total energy density \bar{E} ; it deviates from \bar{E} with increasing amplitude. The amplitudes that can be experimentally generated in liquids are always "small" amplitudes; therefore the statement: radiation pressure = mean energy density holds always for this case in liquids. For other media, as gases, it is also correct for small amplitudes, as shown in Part III, Sec. 2.

b. Now let us consider the case in which the device D is not a perfect absorber, but reflects partially or totally the incident wave and absorbs all that is not reflected. Here the incident and the reflected waves interfere and cause a periodic variation in excess-pressure \bar{p} as well as in $\overline{\rho u^2}$ along the axis of the beam (see Eqs. (2.43) and (2.44), and Sec. II, 7b). In a

beam of infinite width, not in communication with the undisturbed part of the medium at the rear of D, these two quantities \bar{p} and $\overline{\rho u^2}$, acting together at the surface of D, are of opposite sign and cancel each other to zero at small amplitudes. But since experimentally the beam is always of finite width and normally surrounded by the undisturbed medium, the mean pressure \bar{p}_t inside the beam is raised, as we saw in Sec. II, 7b, up to the static pressure p_0 . Superposed upon this mean pressure p_0 is the periodic variation \bar{p} in space; it amounts to $\bar{p}_t - p_0$ and is different from p_0 in general at D. The average negative value of $\bar{p}_t - p_0$ (which in the infinitely extended beam neutralized the quantity $\overline{\rho u^2}$ at D) is now, in the case of the finite beam, increased by the amount $\bar{E} (1 + \gamma^2)$ to bring its value up to p_0 ; $\bar{p}_t - p_0$ no longer neutralizes the action of the flux of momentum $\overline{\rho u^2}$.

Let us write again the expressions for \bar{p}_t and $\overline{\rho u^2}$ in a plane wave to see what happens:

$$\bar{p}_t = \bar{p} + p_0 = -\bar{E} (1 + 2\gamma \cos(2kx + \theta) + \gamma^2) + p_0 \quad (2.43)$$

$$\overline{\rho u^2} = +\bar{E} (1 + 2\gamma \cos(2kx + \theta) + \gamma^2) \quad (2.44)$$

In the case of a perfect absorber D, $\gamma = 0$ and there is no variation of \bar{p}_t and $\overline{\rho u^2}$ with x . The pressure $\bar{p}_t = p_0 - \bar{E}$ is raised to p_0 in the beam and all the force on D is caused by the quantity $\overline{\rho u^2}$ in the way we pointed out above under α . If we consider on the other hand the case of a perfect and rigid reflector D, there is no movement of particles at the surface of D; thus if the origin $x = 0$ is taken at the mean position of the surface of D,

both u and ξ vanish at $x = 0$. This boundary condition requires that $\gamma = 1$ and $\theta = \pi$, as is seen from Eq. (2.24); from Eq. (2.44) above we find $\overline{\rho u^2} = 0$ and $\bar{p}_t = p_0$ at $x = 0$. The first term $\overline{\rho u^2}$ is zero, because there is no motion at $x = 0$; the second term $\bar{p}_t = p_0$ because \bar{p} varies purely sinusoidally in time at $x = 0$ around the average value p_0 . But there is now the additional effect of the surrounding medium, which raises the average value in space of p_t to p_0 as illustrated in Fig. 14.

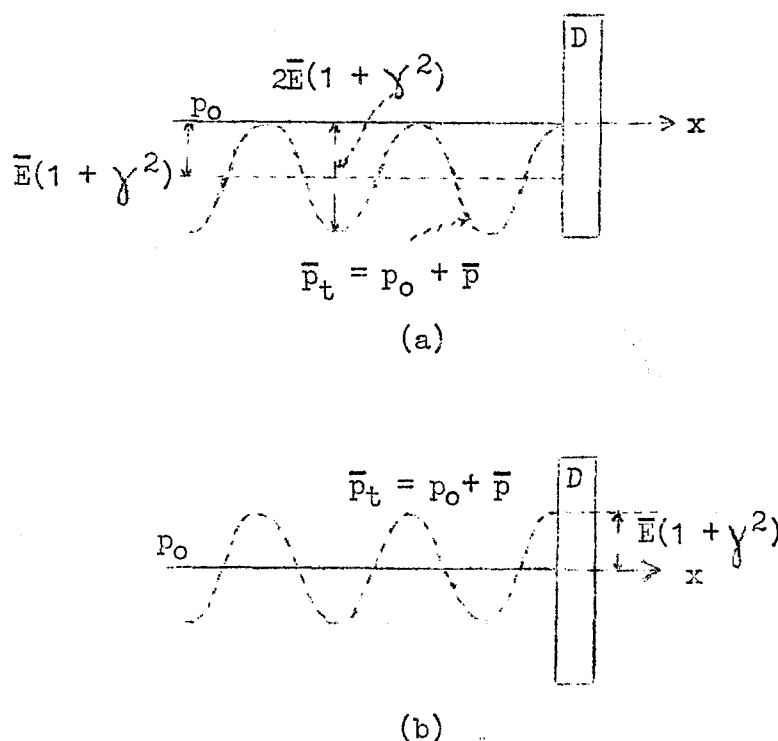


Fig. 14

Time-average excess pressure $\bar{p}(x)$ along the x -axis in a liquid, when the acoustic beam striking the device D undergoes perfect reflection at D ($\gamma = 1$). (a) Beam of infinite width and not in communication with undisturbed regions of liquid. The average value in space of \bar{p}_t is lower than p_0 by the amount $\bar{E}(1 + \gamma^2) = 2\bar{E}$. The value of \bar{p} at $x = 0$ is zero. (b) Beam surrounded by or in communication with undisturbed regions of liquid. The average value in space of \bar{p}_t is raised by $\bar{E}(1 + \gamma^2) = 2\bar{E}$ and therefore equal to p_0 . The value of \bar{p} at $x = 0$ is now $\bar{E}(1 + \gamma^2) = 2\bar{E}$.

The amount by which \bar{p}_t is raised, to bring its average value in space to p_o , is $+\bar{E}(1+\gamma^2)$, as we found already in Sec. II, 7b. At the front of D appears now a mean pressure $(p_t)_D = p_o + \bar{E}(1+\gamma^2)$, or, with $\gamma = 1$ for a perfect reflector, $(p_t)_D = p_o + 2\bar{E}$. At the rear of D, the pressure is p_o . Consequently the force exerted per unit area of the receiving part of D equals $2\bar{E}$, as expected. In the case of a perfect and rigid reflector the radiation pressure measured experimentally is due therefore to a mean excess pressure at the receiving plane rather than to the quantity $\overline{\rho u^2}$, which is zero.

c. In the general case, where the incident energy is partially absorbed and partially reflected, the force exerted upon D is due to both effects; the quantity $\overline{\rho u^2}$ and the mean hydrostatic excess pressure $\bar{p}_t - p_o$. Their sum always amounts to $\bar{E}(1+\gamma^2)$, but the amount contributed by each component depends on the reflection coefficient γ and the phase angle θ of the reflected wave. From Eqs. (2.43) and (2.44) above we find the following values for $P = \bar{p} + \bar{E}(1+\gamma^2)$ and $\overline{\rho u^2}$ at D, where $x = 0$:

$$\bar{P}_{x=0} = -2\gamma\bar{E}\cos\theta + p_o \quad (2.76)$$

$$\begin{aligned} \overline{\rho u^2}_{x=0} &= +2\gamma\bar{E}\cos\theta + \bar{E}(1+\gamma^2) \\ \text{or} \quad \overline{\rho u^2}_{x=0} &= \bar{E}(1+\gamma)^2 - 4\gamma\bar{E}\sin^2\frac{\theta}{2} \end{aligned} \quad (2.77)$$

For the two special cases first considered above, we find here again the previous results: At a perfect absorber with $\gamma = 0$

we get $\bar{P}_{x=0} = p_0$ and $\overline{\rho u_{x=0}^2} = \bar{E}$, the whole force being due to flux of momentum $\overline{\rho u_{x=0}^2}$. At a perfect rigid reflector with $\gamma = 1$ and $\theta = \pi$ we obtain $\bar{P}_{x=0} = p_0 + 2\bar{E}$ and $\overline{\rho u_{x=0}^2} = 0$, the whole force being due to the mean excess pressure $\bar{P}_{x=0}$. The foregoing equations give the contributions of $\bar{P}_{x=0}$ and $\overline{\rho u_{x=0}^2}$ to the resultant force $p_0 + \bar{E}(1 + \gamma^2)$ per unit area of the receiver for all values of γ and θ .

Thus, in the general case, where a reflected wave exists, one has strictly to recognize these two different effects acting simultaneously at a surface struck by a compressional wave.

What is measured experimentally as radiation pressure is the sum of the flux of momentum and the excess pressure at the receiving area, which amounts always to $\bar{E}(1 + \gamma^2)$. The two terms of the sum depend on the reflection coefficient γ and the phase angle θ of the reflected wave; but their sum is independent of θ and simply equals the sum of the incident energy density \bar{E} and the energy density $\gamma^2 \bar{E}$ of the reflected wave. This justifies, so far as numerical results are concerned, the frequently adopted point of view, which regards the incident wave energy as perfectly absorbed by the receiver, leading to a contribution \bar{E} , and then partially or totally reemitted by the receiver, leading to an additional force $\gamma^2 \bar{E}$. This concept gives the right numerical value for the radiation pressure; it does not provide however the physical background for the forces really acting at the receiver. (See also p. 49)

In the case where the acoustic radiation traverses a plane reflector separating two different media 1 and 2, some of the

energy being transmitted into medium 2 behind the reflector, the resultant radiation pressure acting upon the reflector at normal incidence is obviously given by

$$\bar{P}_{\text{rad}} = \bar{p}_{t1} - \bar{p}_{t2} + \overline{\rho_1 u_1^2} - \overline{\rho_2 u_2^2} \quad (2.78)$$

If the media 1 and 2 are under the same hydrostatic pressure p_0 , then $\bar{p}_{t1} - \bar{p}_{t2} = \bar{p}_1 - \bar{p}_2$. For small amplitudes and a beam of finite width, we can write $\bar{p}_1 + \overline{\rho_1 u_1^2} = \bar{E}_1 (1 + \gamma^2)$; assuming that no reflected wave exists in medium 2, we may introduce the amplitude transmission coefficient δ by $\bar{E}_2 = \overline{\rho_2 u_2^2} = \delta^2 \bar{E}_1$ and obtain

$$\bar{P}_{\text{rad}} = \bar{E}_1 (1 + \gamma^2 - \delta^2) \quad (2.79)$$

as already stated on page 53. If the reflector does not absorb any energy, $(1 - \gamma^2) c_1 \bar{E}_1 = \delta^2 c_2 \bar{E}_1$ or $\delta^2 = \frac{c_1}{c_2} (1 - \gamma^2)$ and we obtain from (2.79)

$$\bar{P}_{\text{rad}} = \bar{E}_1 \left[\left(1 - \frac{c_1}{c_2}\right) + \gamma^2 \left(1 + \frac{c_1}{c_2}\right) \right] \quad (2.80)$$

This shows that the direction of the radiation pressure can also reverse its sign if $\gamma^2 < \frac{c_1 - c_2}{c_1 + c_2}$ and $c_1 > c_2$.* No resultant radiation pressure would be found if $\gamma^2 = \frac{c_1 - c_2}{c_1 + c_2}$. If the two media are substantially the same, $c_1 = c_2$, and from Eq. (2.80) we have

$$\bar{P}_{\text{rad}} = 2 \gamma^2 \bar{E} \quad (2.81)$$

* See, for example, the experiments of Hertz and Mende⁸.

PART III

RADIATION PRESSURE IN GASES

1. The Lagrangian Wave Equation for Gases, Neglecting Absorption

The Lagrangian wave equation, which is linear in the case of liquids due to the assumption of constant compressibility, becomes non-linear for gases, as the adiabatic relation between total pressure p_t^* and density ρ^* has the well-known form $p_t^* \cdot V^{\gamma_c} = \text{constant}$, or

$$\frac{p_t^*}{p_o} = \left[\frac{v_o}{V^*} \right]^{\gamma_c} = \left[\frac{p^*}{p_o} \right]^{\gamma_c} \quad (3.1)$$

where γ_c is the ratio of the specific heats. (This notation is chosen in order to avoid confusion with the coefficient of reflection γ). Inserting Eq. (3.1) in the Lagrangian wave equation (2.19), considering $T_{xx}^* = -p_t^*$ and Eq. (2.28) we get, disregarding viscous absorption

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2} \cdot \frac{1}{\left(1 + \frac{\partial \xi}{\partial x}\right)^{1 + \gamma_c}} \quad (3.2)$$

with $c^2 = p_o \gamma_c / \rho_o$.

This equation cannot be solved exactly by known functions in a closed form. So one usually is content to treat only the case of small amplitudes, using the Taylor development of the term

$\left(1 + \frac{\partial \xi}{\partial x}\right)^{-(1 + \gamma_c)}$ and retaining first order terms. In dealing with average time-values, however, one must in general be careful not to omit second order terms. The flux of momentum $\overline{\rho u^2}$, as

we have learned, involves no second order terms of Eq. (3.2) at small amplitudes according to our considerations leading to Eq. (2.37). On the other hand, the correct calculation of the "Rayleigh" pressure $\overline{p^*}$ requires the solution of (3.2) up to second order terms.

2. The Radiation Pressure upon a Perfect Absorber
in Gases at Small Amplitudes due to a Beam of
Finite Width

Developing the right side of Eq. (3.2) in a Taylor series and retaining only second order terms, one finds:

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2} - c^2 (1 + \gamma_c) \frac{\partial \xi}{\partial x} \frac{\partial^2 \xi}{\partial x^2} \quad (3.3)$$

Airy has given the solution of Eq. (3.3) for small amplitudes ξ_0 ($k\xi_0 < 1$) in the form of a series in powers of $k\xi_0$ for a pure sinusoidal excitation assumed to be located at the origin, where $x = 0$; retaining second order terms, it is known to have the following form for a wave traveling in the positive x-direction (see ref. 9, p. 130):

$$\xi = \xi_0 \sin (\omega t - kx) + \frac{(1 + \gamma_c) x}{8} (k\xi_0)^2 [1 + \cos 2 (\omega t - kx)] \quad (3.4)$$

For computing the radiation pressure of a beam of finite width upon a perfect absorber, we have from Part II, 7a,

$$\overline{P}_{\text{rad}} = \overline{\rho u^2} \quad (3.5)$$

Only the first-order term in Eq. (3.4) is needed for computing $\overline{P}_{\text{rad}}$ at small amplitudes. To this approximation $\overline{\rho u^2}$ becomes identical with $\overline{\rho_0 u^{*2}}$ and we find easily from Eq. (3.4)

$$\bar{P}_{\text{rad}} = \frac{\rho_0 \omega^2 \xi_0^2}{2} = 2 \bar{E}_{\text{kin}} \quad (3.6)$$

At small amplitudes in gases the mean kinetic energy density equals the mean potential energy density.* Therefore

$2 \bar{E}_{\text{kin}} = 2 \bar{E}_{\text{pot}} = \bar{E} = \text{mean total energy density.}$ Thus we obtain at small amplitudes in gases the same result that we found in liquids:

$$\bar{P}_{\text{rad}} = \bar{E} \quad (3.7)$$

The radiation pressure upon a perfect absorber in gases for an acoustic beam of finite width equals the energy density for small amplitudes. For larger amplitudes we have to use Eq. (3.5).

The solution for finite amplitudes in gases is rather complicated, as it requires a solution of the non-linear differential equation (3.2), which can only be given by a series development.⁷

Its consideration is beyond the scope of the present paper.

3. The Rayleigh Pressure in Gases for Progressive Waves and Small Amplitudes

For computing the mean value \bar{p}_t^* which would be observed at a moving particle, we start from Eq. (3.1) and obtain by replacing $(\rho^*/\rho_0) \gamma_c$ according to Eq. (2.28)

$$p_t^* = p_0 \left(1 + \frac{\partial \xi}{\partial x} \right) - \gamma_c \quad (3.8)$$

Developing this relation in a Taylor series, and retaining terms up to the second order, we have

$$p_t^* = p_0 \left(1 - \gamma_c \frac{\partial \xi}{\partial x} + \frac{\gamma_c (1 + \gamma_c)}{2} \left[\frac{\partial \xi}{\partial x} \right]^2 \pm \dots \right) \quad (3.9)$$

* See for instance ref. 12, p. 223

Using the solution Eq. (3.4) for ξ in the abbreviated form

$$\xi = h_1 \cdot (k\xi_0) + h_2 \cdot (k\xi_0)^2 + \dots$$

we get

$$p_t^* = p_0 \left[1 - \gamma_c (k\xi_0) \frac{\partial h_1}{\partial x} + (k\xi_0)^2 \left(\frac{\gamma_c (1 + \gamma_c)}{2} \left(\frac{\partial h_1}{\partial x} \right)^2 - \gamma_c \frac{\partial h_2}{\partial x} \right) \right] \quad (3.10)$$

From Eq. (3.10) it is evident that we need the solution for ξ up to the second order terms, which involves the term in h_2 , if we wish to compute $\overline{p_t^*}$ correctly. It is the term $\gamma_c \cdot \partial h_2 / \partial x$ which was omitted by Rayleigh.

Inserting h_1 and h_2 which are given immediately by Eq. (3.4), into Eq. (3.10), and taking the time-average value for p_t^* , we get

$$\overline{p_t^*} = p_0 \left[1 + (k\xi_0)^2 \left(\frac{\gamma_c (1 + \gamma_c)}{4} - \frac{\gamma_c (1_c + \gamma_c)}{8} \right) \right] \quad (3.11)$$

whence

$$\overline{p_t^*} = p_0 \frac{\gamma_c (1 + \gamma_c)}{8} (k\xi_0)^2 + p_0 \quad (3.12)$$

Introducing $\rho_0 = \gamma p_0 / c^2$, $k = \omega / c$, and the mean total energy density $\overline{E} = 2 \overline{E}_{\text{kin}} \omega^2 \rho_0 \xi_0^2 / 2$, we obtain for the Rayleigh pressure* in a progressive wave

$$\overline{P}_{\text{Rayleigh}} = \overline{p_t^*} - p_0 = \frac{1 + \gamma_c}{4} \overline{E} \quad (3.13)$$

This formula differs from Rayleigh's well-known result $(1_c + \gamma_c) \overline{E} / 2$.

* This is in accordance with results obtained also by P. J. Westervelt¹⁶ and J. S. Mendousse¹¹; both authors however did not proceed to the final expression for $\overline{p_t^*} - p_0$, as given in Eq. (3.13) here.

The reason is clearly seen from Eq. (3.11), in which the term $-\gamma_c(1 + \gamma_c)/8$ is due to the fact that the second-order term in Eq. (3.4) is taken into account. Thus Rayleigh's formula must be multiplied by the factor 1/2, to be correct in the case of progressive waves. The remark may be added that the formula (3.13) holds only at not too large distances x from the pure sinusoidal source of excitation. The second term of the solution (3.4) contains the factor x . This corresponds to the known fact that a pure sinusoidal wave in gases is distorted with increasing distance from the source, the wave energy being transferred more and more from the original frequency into higher harmonics. Only in the immediate neighborhood of $x = 0$ is it permissible to define energy density by the expression we used for \bar{E} and therefore to attribute a reasonable physical meaning to the relation (3.13) as well as to the result in Eq. (3.7) for the radiation pressure.

4. The Mean Pressure \bar{p} in Gases for Progressive Waves and Small Amplitudes

Under the same limitations just mentioned, we can give the expression for the mean pressure \bar{p}_t in Eulerian coordinates in a gas traversed by a plane infinitely extended compressional wave. From the generally valid Eq. (2.75) we have in a progressive wave $\bar{p} = \bar{p}^* - \bar{E}$. Introducing Eq. (3.13) and adding the hydrostatic pressure p_0 , we find*

$$\bar{p}_t = p_0 + \frac{1 + \gamma_c}{4} \bar{E} - \bar{E}$$

or

$$\bar{p}_t = p_0 - \frac{3 - \gamma_c}{4} \bar{E} \quad (3.14)$$

* See also P. J. Westervelt, ref. 16, Eq. (29)

Thus, in gases, the terms \bar{p} and $\overline{\rho u^2}$ do not cancel each other in the tensor component $\bar{T}_{xx} = \bar{p}_t + \overline{\rho u^2}$, as they do in a liquid with constant compressibility. In gases we have in the absence of a reflected wave

$$\bar{T}_{xx} = \bar{p}_t + \overline{\rho u^2} = p_o + \frac{1 + \gamma_c}{4} \bar{E} \quad (3.15)$$

5. The Radiation Pressure at a Perfect Absorber in Gases for a Plane Infinitely Extended Wave at Small Amplitudes

For an infinitely extended plane wave the radiation pressure is given by $\bar{P}_{rad} = \bar{p} + \overline{\rho u^2}$, which at small amplitudes equals $\bar{p} + \bar{E}$. Since for a progressive wave $\bar{p} = \bar{p}^* - \bar{E}$ according to Eq. (2.75), we find in this case

$$\bar{P}_{rad} = \bar{p}^* = \bar{P}_{Rayleigh} = \frac{1 + \gamma_c}{4} \bar{E} \quad (3.16)$$

The Rayleigh pressure (in the corrected form for progressive waves) becomes therefore identical with the true radiation pressure in a beam of infinite width, which is not in communication with a gas region unaffected by the wave motion.

It would be very difficult to realize this case by experiment, even approximately; we usually have to deal with a beam of finite cross section and in communication with an undisturbed gas region. But in the latter case the mean pressure \bar{p}_t in Eq. (3.14) is raised to p_o and the measured radiation pressure becomes identical with $\overline{\rho u^2} = \bar{E}$, as shown in Eq. (3.7).

6. Mean Density in Progressive Waves in Gases at Small Amplitudes

A brief treatment of this subject is given, especially

to point out the difference between liquids and gases.

The time average value of the change in density $\overline{\Delta \rho}$ can be derived from Eq. (2.28). At small amplitudes we have

$$\frac{\rho^*}{\rho_0} = (1 + \frac{\partial \xi}{\partial x})^{-1} = 1 - \frac{\partial \xi}{\partial x} + \left(\frac{\partial \xi}{\partial x}\right)^2 + \dots \quad (3.17)$$

The solution for ξ up to second order terms is given by (3.4). Inserting this solution in (3.17) and retaining all second order terms leads to the expression for the density ρ^* in Lagrangian coordinates. The change in density $\Delta \rho = \rho - \rho_0$ in Eulerian coordinates is found by substituting $x - \xi$ for x , according to Eq. (2.21). Doing so, developing the result in powers of $k\xi_0$, retaining second order terms, and finally averaging in time, we find

$$\frac{\Delta \rho}{\rho_0} = \frac{\bar{\rho} - \rho_0}{\rho_0} = -\frac{1 + \gamma_0}{8} (k\xi_0)^2 \quad (3.18)$$

The mean density in an infinitely extended plane progressive wave in gases is lowered by the amount given in (3.18), that is by a term as low as the second order in $k\xi_0$. In liquids of constant compressibility, on the other hand, we found that the change in average density vanished in the second order approximation, as it is only of fourth order in $k\xi_0$. (See Eq. (2.68a), p. 63).

The mean pressure \bar{p}_t in a progressive wave is lowered by the wave motion by the amount given in Eq. (3.14). In a beam of finite width surrounded by an undisturbed medium of hydrostatic pressure p_0 the beam undergoes a slight compression until the pressure inside the beam equals p_0 , as pointed out already in the case of a liquid medium (Part II, Sec. 7a). The increase in density $\Delta \rho'$ inside the beam due to this compression is given by

$$\frac{\Delta \rho'}{\rho_0} = \frac{\rho - \rho_0}{\rho_0} = \left(\frac{p_0 + \Delta p}{p_0} \right)^{1/\gamma_c} - 1 = \frac{1}{\gamma_c} \frac{\Delta p}{p_0} + \dots$$

whence, according to Eq. (3.14), we find

$$\overline{\frac{\Delta \rho'}{\rho_0}} = \frac{3 - \gamma_c}{4} \frac{\bar{E}}{\rho_0 c^2} = \frac{3 - \gamma_c}{8} (k \xi_0)^2 \quad (3.19)$$

The resultant change in density in a beam of finite width in a progressive wave in a gas at small amplitudes is therefore given by means of Eqs. (3.18) and (3.19) as

$$\overline{\Delta \rho_{\text{total}}} = \overline{\Delta \rho} + \overline{\Delta \rho'} = \frac{1 - \gamma_c}{2} \frac{\bar{E}}{c^2} = \rho_0 \frac{1 - \gamma_c}{4} (k \xi_0)^2 \quad (3.20)$$

7. Radiation Pressure in Gases at a Perfect Reflector due to a Finite Beam at Small Amplitudes

The boundary condition at a perfect and rigid plane reflector is by definition $\xi = 0$ or $u^* = u = 0$. The corresponding first-order solution of the wave equation (3.2) or (3.3), which we denote by $\xi^{(1)}$ and $u^{*(1)}$ respectively, is

$$\xi^{(1)} = \xi_0 (\sin(\omega t - kx) - \sin(\omega t + kx))$$

$$\text{or} \quad \xi^{(1)} = 2\xi_0 \sin kx \cos(\omega t + \pi) \quad (3.21)$$

$$\text{and} \quad u^{*(1)} = \frac{\partial \xi}{\partial t} = 2\omega \xi_0 \sin kx \sin \omega t \quad (3.22)$$

the well-known expressions for a standing wave; it fulfills the boundary condition $u = 0$ at any time, the reflector being, for example, at $x = 0$. From (3.22) we obtain the mean kinetic energy density, averaged in time and space, by

$$2 \bar{E}_{\text{kin}} = \rho_0 \overline{u^{*2}} = \rho_0 \overline{u^2} = \rho_0 \omega^2 \xi_0^2 \quad (3.23)$$

This expression is correct up to the second order of approximation as already stated on Page 35.

If we apply the method mentioned by L. Brillouin and outlined on Page 49 of this report for computing the radiation pressure upon a reflector, we have to add the radiation pressure

$\gamma^2 \cdot \frac{\rho_0 \omega^2 \xi_0^2}{2}$, regarded as reemitted by a reflector of amplitude reflection coefficient γ , to the radiation pressure $\frac{\rho_0 \omega^2 \xi_0^2}{2}$, found in Eq. (3.6) as due to an incident and perfectly absorbed plane wave. The sum results in

$$\bar{P}_{\text{rad}} = (1 + \gamma^2) \cdot \frac{\rho_0 \omega^2 \xi_0^2}{2} \quad (3.24)$$

and for the perfect reflector with $\gamma = 1$ we obtain

$$\bar{P}_{\text{rad}} \big|_{\gamma=1} = \rho_0 \omega^2 \xi_0^2 = 2 \bar{E}_{\text{kin}} = 2 \bar{E} \quad (3.25)$$

where \bar{E} denotes the mean total energy density of each of the two progressive wave components leading to Eq. (3.21), in conformity with the usage in this report, according to Eq. (2.42).

The problem of a perfect and rigid reflector was also treated by F. Dopp³. It may contribute to one's confidence in the result expressed by Eq. (3.25), if we trace also Bopp's reasoning for this case, especially as it can be done very briefly. Averaging in time the generally valid Eulerian equation (2.14), we have for a wave motion periodic in time $\partial (\bar{p} + \overline{\rho u^2}) / \partial x = 0$, and integrating with respect to x , we find

$$\bar{p} + \overline{\rho u^2} = C \quad (3.26)$$

At the perfect reflector $u = 0$; thus the constant C equals the mean excess pressure \bar{p} at the reflector, which is just identical with the radiation pressure $(\bar{P}_{\text{rad}})_{\gamma=1}$ in our case. (See Part II, Sec. 12b). On the other hand we obtain by averaging Eq. (3.22) in space

$$\bar{p} + \overline{\rho u^2} = C = \bar{p} + 2 \bar{E}_{\text{kin}} \quad (3.27)$$

For the beam of finite width under action of the static pressure p_0 of the surrounding medium it is reasonable to assume, in agreement with Bopp, that $\bar{p} = 0$ (See also Part II, Sec. 7b), in other words that the average value in time and space of the total pressure p_t inside the beam equals the static pressure p_0 outside the beam. With $\bar{p} = 0$ we get from (3.27) $C = (\bar{P}_{\text{rad}})_{\gamma=1} = 2 \bar{E}_{\text{kin}}$, in agreement with Eq. (3.25). (See also Page 92, Eq. (3.41))

8. The Radiation Pressure in Gases at a Perfect Reflector Caused by an Infinitely Extended Plane Wave at Small Amplitudes

The solution of this problem encounters some difficulties, as it turns out that the complete solution for a standing wave in gases cannot be achieved without taking into account the dissipation of energy, which actually is of course always present. Nevertheless it is possible to compute the radiation pressure for this case without proceeding to the complete solution, as will now be shown.*

The radiation pressure at the perfect reflector becomes identical with

$$(\bar{P}_{\text{rad}})_{\gamma=1} = (\bar{p}_t^* - p_0 = \bar{p}^* = \bar{p})_{x=0} \quad (3.28)$$

* See also Westervelt's paper, ref. 16, Sec. VI.

since all these pressures are the same at the immovable surface of the reflector. The time average value of p_t^* is given by Eq. (3.10):

$$\overline{p_t^*} = p_0 \left[1 - \gamma_c(k\xi_0) \frac{\partial h_1}{\partial x} + \gamma_c(k\xi_0)^2 \left\{ \frac{1 + \gamma_c}{2} \left(\frac{\partial h_1}{\partial x} \right)^2 - \frac{\partial h_2}{\partial x} \right\} \right] \quad (3.29)$$

From the first order solution $\xi^{(1)}$ in (3.21) we find easily

$$\left(\frac{\partial h_1}{\partial x} \right) = 0 \quad \text{and} \quad \left(\frac{\partial h_1}{\partial x_1} \right)^2 = 1 + \cos 2 kx \quad (3.30)$$

To find the term $\partial h_2 / \partial x$, which is related to the second order term $\xi^{(2)}$ of $\xi = \xi^{(1)} + \xi^{(2)}$, we need the solution of the differential equation (3.3), which in this case, introducing the first order solution $\xi^{(1)}$ into the last term on the right side, becomes

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2} + 2 c^2 (1 + \gamma_c) k^3 \xi_0^2 \sin 2 kx \cos^2 \omega t \quad (3.31)$$

A solution compatible with the boundary condition $\xi = 0$ at $x = 0$ at any t , is found to have the form¹⁶

$$\xi = \xi^{(1)} + \xi^{(2)} = \xi^{(1)} + f(t) \cdot (k\xi_0)^2 \sin 2 kx \quad (3.32)$$

Inserting (3.32) into (3.31) with $\xi^{(1)}$ given by (3.21), we obtain for the time-dependent function $f(t)$ the differential equation

$$\frac{d^2 f}{dt^2} + 4 \omega^2 f = (1 + \gamma_c) \omega c (1 + \cos 2 \omega t) \quad (3.33)$$

This is the differential equation of a resonant system driven by an impressed force of exactly the resonance frequency 2ω . As there is no dissipation term involved on the left side of (3.33), the final amplitude becomes infinite, as is well known. This

fact can easily be understood: Due to the non-linear relation between pressure and density in gases, the fundamental angular frequency ω , which excites the standing wave, sets up higher harmonics of angular frequencies 2ω , 3ω , ... in the system. The amplitudes of all the harmonics are limited actually by the unavoidable dissipation of energy in the resonating system, which however is not taken into account in the differential equation (3.31). A solution leading to finite amplitudes can only be achieved if dissipation or absorption of energy is properly taken into consideration.

In our problem, however, we are only interested in $\overline{p_t^*}$, which involves time-average values of $\xi^{(2)}$: Fortunately the time-average values of all those purely sinusoidal higher harmonics vanish; the only term of $f(t)$ in the differential equation (3.33) that is of importance for us here is the constant term $(1 + \gamma_c) \omega c$ on the right side of (3.33), which leads to a time-independent term

$$f = \frac{(1 + \gamma_c) c}{4 \omega} = \frac{1 + \gamma_c}{4 k} \quad (3.34)$$

With the aid of (3.32) we have therefore

$$\overline{\xi^{(2)}} = (k\xi_0)^2 \overline{h_2(x)} = (k\xi_0)^2 \frac{1 + \gamma_c}{4 k} \sin 2 kx \quad (3.35)$$

$$\text{and hence} \quad \frac{\partial h_2}{\partial x} = \frac{1 + \gamma_c}{2} \cos 2 kx \quad (3.36)$$

From Eqs. (3.30), (3.36) and (3.29) we obtain

$$\overline{p_t^*} - p_0 = \frac{p_0 \gamma_c (1 + \gamma_c)}{2} (k\xi_0)^2 (1 + \cos 2 kx - \cos 2 kx)$$

$$\text{or} \quad \overline{p^*} = \frac{\rho_0 \omega^2 \xi_0^2}{2} (1 + \gamma_c) \quad (3.37)$$

Taking account of (3.23) and (3.28), we have finally

$$(\bar{P}_{\text{rad}})_{\gamma=1} = (\bar{p}^*)_{x=0} = (1 + \gamma_c) \bar{E}_{\text{kin}} = (1 + \gamma_c) \bar{E} \quad (3.38)$$

It is interesting to note that the "Rayleigh pressure" \bar{p}^* , as observed at a moving particle, becomes independent of x in a standing wave.

From (3.37) we are able to derive the time-average value of the excess pressure \bar{p} in Eulerian coordinates; with the aid of Eqs. (2.74) and (3.18) we find

$$\begin{aligned} \bar{p} &= \bar{p}^* - 2 \bar{E}_{\text{kin}}(x) = \rho_0 \omega^2 \xi_0^2 \left(\frac{1 + \gamma_c}{2} - 1 + \cos 2 kx \right) \\ \text{or} \quad \bar{p} &= \bar{E}_{\text{kin}} (\gamma_c - 1 + 2 \cos 2 kx) \end{aligned} \quad (3.39)$$

The average value in time and space of \bar{p} is therefore

$$\bar{\bar{p}} = (\gamma_c - 1) \bar{\bar{E}}_{\text{kin}} \quad (3.40)$$

The mean change in pressure due to the wave motion is positive in gases, since $\gamma_c > 1$. If the beam is surrounded by an undisturbed medium with hydrostatic pressure p_0 , it tends to expand a little, until the average total pressure $\bar{\bar{p}}_t$ equals the hydrostatic pressure p_0 outside the beam. In the beam of finite width the pressure \bar{p} becomes therefore according to (3.39)

$$\bar{p} = 2 \bar{\bar{E}}_{\text{kin}} \cos 2 kx \quad (3.41)$$

At the reflector, where $x = 0$, we have then $(\bar{p})_{x=0} = 2 \bar{\bar{E}}_{\text{kin}}$; this leads us back, by a different route, to the result found in (3.25) for the radiation pressure in a beam of finite width.

9. Note Concerning Rayleigh's Original Formula
on the "Pressure of Vibrations"

Rayleigh's original formula $\frac{1 + \gamma_c}{2} \overline{(\text{energy density})}^{13}$ is found in many books and publications as the formula for radiation pressure in gases. Nevertheless it does not express the radiation pressure as usually measured in a beam of finite width, but is rather associated with the mean excess pressure $\overline{p^*} = \overline{p_t^*} - p_0$ observed at a moving particle, which may properly be called the "Rayleigh pressure". On the other hand we found in Sec. 3 and § above the following results for the radiation pressure in an infinitely extended plane wave in gases at small amplitudes:

$$(\overline{P}_{\text{rad}})_{\gamma=0} = \frac{1 + \gamma_c}{4} \overline{E} = \frac{1 + \gamma_c}{4} \cdot \frac{\rho_0 \omega^2 \xi_0^2}{2} \quad (3.13)$$

$$(\overline{P}_{\text{rad}})_{\gamma=1} = (1 + \gamma_c) \overline{E} = (1 + \gamma_c) \frac{\rho_0 \omega^2 \xi_0^2}{2} \quad (3.38)$$

One must be careful as to the meaning of energy density; throughout this report \overline{E} denotes the mean energy density of a progressive wave. In the formula (3.13) \overline{E} becomes identical with the actual total energy density for a progressive wave. In the formula (3.38) the actual total energy density equals $2 \overline{E}$ because the energy density of the reflected wave adds to the energy density of the wave incident on the perfect reflector. If we introduce the concept of the total energy density $\overline{E}_{\text{total}}$, the two formulas may be written in the form

$$(\overline{P}_{\text{rad}})_{\gamma=0} = \frac{1 + \gamma_c}{4} \overline{E}_{\text{total}} \quad (3.13)$$

$$(\overline{P}_{\text{rad}})_{\gamma=1} = \frac{1 + \gamma_c}{2} \overline{E}_{\text{total}} \quad (3.38)$$

This shows that Rayleigh's formula in its original form applies indeed to the radiation pressure on a perfect and rigid reflector in a gas exerted by an infinitely extended plane wave, which does not communicate with a medium under a constant hydrostatic pressure p_0 if the total mean energy density $\bar{\bar{E}}_{\text{total}}$ means the whole energy density $\bar{\bar{E}}_{\text{kin}} + \bar{\bar{E}}_{\text{pot}} = 2 \bar{\bar{E}}_{\text{kin}}$ in the standing wave. For a progressive wave falling upon a perfect absorber, however, the correct value is half as large, if by "total mean energy density", $\bar{\bar{E}}_{\text{total}}$, one understands the total energy density $\bar{\bar{E}}_{\text{kin}} + \bar{\bar{E}}_{\text{pot}} = \bar{\bar{E}}_{\text{kin}} + \bar{\bar{E}}_{\text{pot}}$ of the progressive wave.

ASSEMBLY OF CHIEF EQUATIONS FOR LIQUIDS AND GASES*

A. General Expressions Valid at Large or Small Amplitudes in Liquids and Gases (Disregarding Viscosity)

When a plane acoustic wave of finite or infinite width in a medium (1) falls normally on a plane slab of any material or thickness, separating medium (1) from some other medium (2), the resultant time-average pressure upon the slab is given by

$$\bar{P} = \bar{p}_{t1} - \bar{p}_{t2} + \overline{\rho_1 u_1^2} - \overline{\rho_2 u_2^2} \quad (2.78)$$

Medium (2) may be bounded by still another medium which causes a reflected wave.

If the two media (1) and (2) are both under the same hydrostatic pressure p_0 when undisturbed, Eq. (2.78) reduces to

* For Symbols see pages ii and iii

$$\bar{P}_{\text{rad}} = \bar{p}_1 - \bar{p}_2 + \overline{\rho_1 u_1^2} - \overline{\rho_2 u_2^2} \quad (2.78a)$$

which now represents radiation pressure alone.

B. No Reflected Wave Present in Medium (2)

1. If no reflected wave is present in medium (2), but the slab still transmits and absorbs, we have for a beam of finite width at small amplitudes and surrounded by undisturbed regions (1) and (2),

$$\bar{P}_{\text{rad}} = (1 + \gamma^2 - \sigma^2) \bar{E}_1 \quad (2.79)$$

2. If no energy is absorbed within the slab, the foregoing formula for a beam of finite width at small amplitudes can be reduced to

$$\bar{P}_{\text{rad}} = \left[\left(1 - \frac{c_1}{c_2}\right) + \gamma^2 \left(1 + \frac{c_1}{c_2}\right) \right] \bar{E}_1 \quad (2.80)$$

C. Opaque Slab, No Wave Motion Present in Medium (2)

1. If the slab absorbs all energy that is not reflected, so that no wave motion is transmitted into medium (2), the radiation pressure exerted by a beam of finite width at small amplitudes in liquids and gases is given by

$$\bar{P}_{\text{rad}} = (1 + \gamma^2) \bar{E}_1 \quad (\text{Eqs. (2.50), (2.54), (2.55), (3.7), (3.24), (3.25)})$$

2. Under the same circumstances, but if the acoustic plane wave is regarded as infinitely extended and not communicating with undisturbed regions, we have for

a. Liquids (with constant compressibility)

$$\bar{P}_{\text{rad}} = 0 \quad (\text{Pages 33, 36, 45})$$

b. Gases (for adiabatic processes)

$$\bar{P}_{\text{rad}} = \frac{1 + \gamma_c}{4} \bar{E}_1 \text{ in a progressive wave} \quad (3.13)$$

$$\bar{P}_{\text{rad}} = (1 + \gamma_c) \bar{E}_1 \text{ in a standing wave} \quad (3.38)$$

The symbol \bar{E} stands for the time average of the energy density of a purely progressive acoustic wave of angular frequency ω and maximal amplitude ξ_0 in a medium of undisturbed density ρ_0 , as given by

$$\bar{E}_1 = \frac{\rho_0 \omega^2 \xi_0^2}{2} = \frac{J}{c} \quad (2.42)$$

where J denotes the intensity of the progressive wave.

D. Mean Excess Pressure Due to a Plane Infinitely Extended Acoustic Wave at Small Amplitudes

The hydrostatic pressure p_0 in the undisturbed medium is changed by the acoustic wave motion into $p_0 + p$, where p is called the excess pressure.

1. Liquids (with constant compressibility)

$$\begin{aligned} \bar{p} &= -\bar{E} (1 + 2\gamma \cos(2kx + \theta) + \gamma^2) \\ \bar{p} &= -\bar{E} (1 + \gamma^2) \end{aligned} \quad (2.43)$$

2. Gases (for adiabatic processes)

$$\bar{p} = -\frac{3 - \gamma_c}{4} \bar{E} \text{ in the progressive wave } (\gamma = 0) \quad (3.14)$$

$$\bar{p} = \bar{E} (\gamma_c - 1 + 2 \cos 2kx) \text{ in the standing wave } (\gamma = 1) \quad (3.39)$$

$$\bar{p} = \bar{E} (\gamma_c - 1) \text{ in the standing wave} \quad (3.40)$$

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